Journal of Geometry and Physics 87 (2015) 266–274

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On the geometry of motions in one integrable problem of the rigid body dynamics☆

I.I. Kharlamova*, A.Y. Savushkin

Russian Academy of National Economy and Public Administration, Volgograd Branch, Volgograd, Gagarin Street 8, Russia

ARTICLE INFO

Article history:
Received 27 November 2013
Received in revised form 26 December 2013
Accepted 3 July 2014
Available online 7 July 2014

MSC:
37J35
70E17

Keywords:
Rigid body
Hodograph method
Partial motions
Decomposition

ABSTRACT

Due to Poinsot's theorem, the motion of a rigid body about a fixed point is represented as rolling without slipping of the moving hodograph of the angular velocity over the fixed one. If the moving hodograph is a closed curve, visualization of motion is obtained by the method of P.V. Kharlamov. For an arbitrary motion in an integrable problem with an axially symmetric force field the moving hodograph densely fills some two-dimensional surface and the fixed one fills a three-dimensional surface. In this paper, we consider the irreducible integrable case in which both hodographs are two-frequency curves. We obtain the equations of bearing surfaces, illustrate the main types of these surfaces. We propose a method of the so-called non-straight geometric interpretation representing the motion of a body as a superposition of two periodic motions.

Introduction

According to the famous result of L. Poinsot [1], an arbitrary motion of a rigid body about a fixed point is represented by rolling without slipping of the moving hodograph of the angular velocity vector over the fixed hodograph of this vector. Since these two curves viewed from the same space have at any time moment the common tangent line, the similar statement is valid also for the conical surfaces generated by the instant rotation axis, namely, the moving axoid is rolling without slipping over the fixed one. The point is that, in the purely rotational motion of a rigid body, the so-called relative and absolute time-derivatives of the angular velocity coincide. The general situation is illustrated in Fig. 1. Here ω is the angular velocity, $\dot{\omega}$ stands for the relative time-derivative (with respect to some reference frame strictly attached to the body) and $d\omega/dt$ is the absolute time-derivative (with respect to an inertial frame).

Estimating his own work, Poinsot wrote that "it enables us to represent to ourselves the motion of a rigid body as clearly as that of a moving point". Due to the problems of finding the fixed hodograph, Poinsot gave only one representation of motion, namely, for the Euler case of the body rotation without external forces.

The classical integrable cases in the rigid body dynamics deal with the motions of a rigid body under the influence of axially symmetric potential fields (the homogeneous gravity field, the central Newtonian field). Such systems are called *reducible* since the corresponding Euler–Poisson equations describe the motion up to unknown rotations about the force symmetry axis and can be reduced to Hamiltonian systems with two degrees of freedom. This means that while the moving hodograph is completely defined by the solution of the Euler–Poisson equations, the equations of the fixed hodograph must

☆ The work is partially supported by RFBR, research project No. 13-01-97025.
* Corresponding author. Tel.: +7 9173320269.
E-mail addresses: xelen28@yandex.ru, irinah@vags.ru (I.I. Kharlamova), sandro@vags.ru (A.Y. Savushkin).

http://dx.doi.org/10.1016/j.geomphys.2014.07.010
0393-0440/

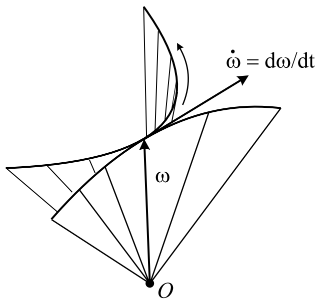

Fig. 1. Poinsot's theorem.

include an additional quadrature. The same problem arises for the solutions of the Kirchhoff equations of rigid body motion in an ideal fluid, which also can be treated as the equations of a gyrostat motion about a fixed point.

P.V. Kharlamov [2] proposed a natural way to find the fixed hodograph and to investigate its properties for all values of the existing parameters. This method is known as the hodographs method of the kinematic interpretation of motion and is based on applying some non-holonomic kinematic characteristics. If the solution of the Euler–Poisson equations is periodic, then the moving hodograph is, obviously, a closed curve. In this case the fixed hodograph, as a rule, densely fills a domain on a two-dimensional surface. P.V. Kharlamov has shown that this surface is a surface of rotation with the meridian completely defined by the initial periodic solution by means of explicit functions and the missing angular coordinate of the fixed hodograph can be found by integrating the known function of time. The equations obtained in [2] gave rise to geometric interpretations built for numerous cases of partial integrability (see reviews in [3,4] and the contemporary state of investigations in [5]).

For an arbitrary motion in integrable reducible systems, the moving hodograph in the generic case is a two-frequency vector function of time. Therefore, the fixed hodograph for almost all initial data densely fills a three-dimensional region in space. In [6–8], I.N. Gashenenko investigated the hodographs properties for quasi-periodic solutions in the classical cases of Goryachev–Chaplygin and Kovalevskaya. He used the equations of the paper [2], the ideas of the number theory and Fourier analysis to describe those classes of non-resonant motions in the integrable reducible rigid body systems for which the coordinates of the angular velocity in the inertial space also are two-periodic functions of time. In other words, on these motions with irrational rotation number on the corresponding regular Liouville torus (on the connected component of a regular integral manifold of the Euler–Poisson equations) both moving and fixed hodographs fill compact two-dimensional surfaces. The motion is represented by rolling of one surface "through" another in such a way that a curve dense in the first surface rolls without slipping over a similar curve dense in the second surface. Still such motions are destroyed by small perturbations of the integral constants.

In this paper, we consider the case when the force field does not have any symmetry axis. The system then cannot be globally reduced to two degrees of freedom. Nevertheless, in integrable systems the motions consisting of the points where the first integrals are dependent play the most important role in the topological analysis of the initial system as a whole, and their geometry can present, clearly enough, the separating cases for different types of the body rotation. Such critical motions are organized into invariant four-dimensional manifolds with the induced dynamics described by Hamiltonian systems with two degrees of freedom called critical subsystems (see e.g. [9]). All regular integral manifolds of the critical subsystems consist of two-dimensional tori (Liouville tori) and both hodographs lie in the two-dimensional surfaces obtained as the images of the corresponding torus under projections from the 6-dimensional phase space onto three-dimensional spaces of the angular velocities (with respect to the rotating body and to some inertial frame). For non-resonant cases the hodographs are dense in these surfaces. The explicit solutions of the critical subsystems allow us to obtain an immediate computer visualization of the surfaces bearing the hodographs. Below we illustrate this process for one of the critical subsystems in the generalized Kovalevskaya case. Simultaneously, we propose another way to describe the body's motion by presenting it as a composition of some simple motions. This composition is based strictly on the known separation of variables.

1. The explicit solution

We now consider the system with two degrees of freedom found in [10]. Its explicit algebraic solution in separated variables and the rough topological analysis are given in [11]. Let us write out the solution in a slightly different form convenient for the purposes of this paper. Suppose that the rigid body with the inertia tensor of the Kovalevskaya type is placed in two linearly independent homogeneous force fields with the centers of application of the fields in the equatorial plane of the body. Let O be the fixed point and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the orthonormal basis of the principal inertia axes. The inertia tensor after choosing the dimensionless values becomes diag{2, 2, 1}. As shown in [9], the forces can always be supposed orthogonal and the centers of application can be taken on the principal inertia axes pointed out from O by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be the unit direction vectors of the intensities of the force fields fixed in the inertial space and represented by their coordinates in the moving frame $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Then the geometric integrals take the form $\boldsymbol{\alpha}^2 = 1, \boldsymbol{\beta}^2 = 1, \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0,$