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Non-integrability vs. integrability in pentagram maps

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ABSTRACT

We revisit recent results on integrable cases for higher-dimensional generalizations of the 2D pentagram map: short-diagonal, dented, deep-dented, and corrugated versions, and define a universal class of pentagram maps, which are proved to possess projective duality. We show that in many cases the pentagram map cannot be included into integrable flows as a time-one map, and discuss how the corresponding notion of discrete integrability can be extended to include jumps between invariant tori. We also present a numerical evidence that certain generalizations of the integrable 2D pentagram map are non-integrable and present a conjecture for a necessary condition of their discrete integrability.

The goal of this paper is three-fold. First we revisit the recent progress in finding integrable generalizations of the 2D pentagram map. Secondly, we discuss a natural framework for the notion of a discrete integrable Hamiltonian map. It turns out that the Arnold–Liouville theorem on existence of invariant tori admits a natural generalization to allow discrete dynamics with jumps between invariant tori, which is relevant for many pentagram maps. Lastly, we define a universal class of pentagram-type maps, describe a projective duality for them, and present a numerical evidence for non-integrability of several pentagram maps in 2D and 3D. In view of many new integrable generalizations found recently, a search for a non-integrable generalization of the pentagram map was brought into light, and the examples presented below might help focusing the efforts for such a search.

1. Types of pentagram maps

Recall that the pentagram map is a map on plane convex polygons considered up to their projective equivalence, where a new polygon is spanned by the shortest diagonals of the initial one, see [1]. It exhibits quasi-periodic behavior of (projective classes of) polygons in 2D under iterations, which indicates hidden integrability. The integrability of this map was proved in [2], see also [3].

While the pentagram map is in a sense unique in 2D, its generalizations to higher dimensions allow more freedom. It turns out that while there seems to be no natural generalization of this map to polyhedra, one can suggest several natural integrable extensions of the pentagram map to the space of generic twisted polygons in higher dimensions.

Definition 1.1. A *twisted n-gon* in a projective space $\mathbb{P}^d$ with a monodromy $M \in SL_{d+1}$ is a doubly-infinite sequence of points $v_k \in \mathbb{P}^d$, $k \in \mathbb{Z}$, such that $v_{k+n} = M \circ v_k$ for each $k \in \mathbb{Z}$, and where M acts naturally on $\mathbb{P}^d$. We assume that the vertices v_k

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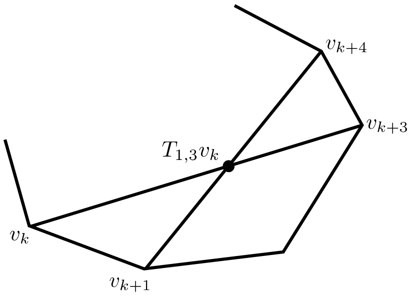

Fig. 1. Deeper pentagram map $T_{1,3}$ in 2D.

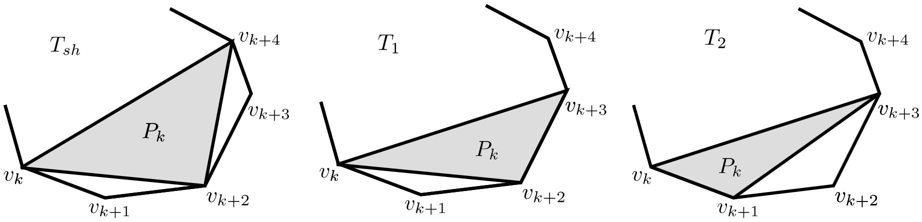

Fig. 2. Different diagonal planes in 3D: for $T_{\rm sh}$, T_1, and T_2.

are in general position (i.e., no $d+1$ consecutive vertices lie in the same hyperplane in $\mathbb{P}^d$), and denote by $\mathcal{P}_n$ the space of generic twisted n-gons considered up to the projective equivalence.

We use projective spaces defined over reals $\mathbb{R}$ (as the easiest ones to visualize), over complex numbers $\mathbb{C}$ (to describe algebraic–geometric integrability), and over rational numbers $\mathbb{Q}$ (to perform a non-integrability test). All definitions below work for any base field. General pentagram maps are defined as follows.

Definition 1.2. We define 3 types of diagonal hyperplanes for a given twisted polygon (v_k) in $\mathbb{P}^d$. (a) The *short-diagonal hyperplane* $P_k^{\rm sh}$ is defined as the hyperplane passing through d vertices of the n-gon by taking every other vertex starting with v_k:

$$P_k^{\rm sh} := (v_k, v_{k+2}, v_{k+4}, \dots, v_{k+2(d-1)}).$$

(b) The *dented diagonal plane hyperplane* P_k^m for a fixed $m = 1, 2, \dots, d-1$ is the hyperplane passing through all vertices from v_k to v_{k+d} but one, by skipping only the vertex v_{k+m}:

$$P_k^m := (v_k, v_{k+1}, \dots, v_{k+m-1}, v_{k+m+1}, v_{k+m+2}, \dots, v_{k+d}).$$

(c) The *deep-dented diagonal plane hyperplane* P_k^m for fixed positive integers m and $p \geq 2$ is the hyperplane as above that passes through consecutive vertices, except for one jump, when it skips $p-1$ vertices $v_{k+m}, \dots, v_{k+m+p-2}$:

$$P_k^{m,p} := (v_k, v_{k+1}, \dots, v_{k+m-1}, v_{k+m+p-1}, v_{k+m+p}, \dots, v_{k+d+p-2}).$$

(Here $P_k^{m,2}$ corresponds to P_k^m in (b).)
Now the corresponding *pentagram maps* $T_{\rm sh}$, T_m, and $T_{m,p}$ are defined on generic twisted polygons (v_k) in $\mathbb{P}^d$ by intersecting d consecutive diagonal hyperplanes:

$$T v_k := P_k \cap P_{k+1} \cap \dots \cap P_{k+d-1},$$

where each of the maps $T_{\rm sh}$, T_m, and $T_{m,p}$ uses the definition of the corresponding hyperplanes $P_k^{\rm sh}$, P_k^m, and $P_k^{m,p}$. These pentagram maps are generically defined on the classes of projective equivalence of twisted polygons $T : \mathcal{P}_n \to \mathcal{P}_n$.

Example 1.3. For $d = 2$ one can have only $m = 1$ and the definitions of $T_{\rm sh}$ and T_m coincide with the standard 2D pentagram map $T_{\rm st}$ in [1] (up to a shift in vertex numbering). The deep-dented maps $T_{1,p}$ in 2D are the maps $T_{1,p}v_k := (v_k, v_{k+p}) \cap (v_{k+1}, v_{k+p+1})$ obtained by intersecting deeper diagonals of twisted polygons, see Fig. 1.

For $d = 3$ the map $T_{\rm sh}$ uses the diagonal planes $P_k^{\rm sh} := (v_k, v_{k+2}, v_{k+4})$, while for the dented maps T_1 and T_2 one has $P_k^1 = (v_k, v_{k+2}, v_{k+3})$ and $P_k^2 = (v_k, v_{k+1}, v_{k+3})$, respectively, see Fig. 2.

Theorem 1.4. *The short-diagonal $T_{\rm sh}$, dented T_m and deep-dented $T_{m,p}$ maps are integrable in any dimension d on both twisted and closed n-gons in a sense that they admit Lax representations with a spectral parameter.*

The integrability of the standard 2D pentagram map $T_{\rm st} := T_{\rm sh} = T_m$ was proved in [2], while its Lax representation was found in [3]. In [4] integrability of the pentagram map for corrugated polygons (which we discuss below) was proved, which implies integrability of the maps $T_{1,p}$ in 2D.