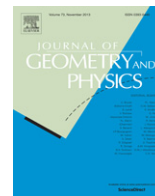




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Integrability in differential coverings



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ABSTRACT

Let $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ be a differential covering of a PDE $\tilde{\mathcal{E}}$ over \mathcal{E} . We prove that if \mathcal{E} possesses infinite number of symmetries and/or conservation laws then $\tilde{\mathcal{E}}$ has similar properties.

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0. Introduction

The notion of a covering (or, better, *differential covering*) was introduced by Vinogradov in [1] and elaborated in detail later in [2,3]. Coverings, explicitly or implicitly, provide an adequate background to deal with nonlocal aspects in the geometry of PDEs (nonlocal symmetries and conservation laws, Wahlquist–Estabrook prolongation structures, Lax pairs, zero-curvature representations, etc.). Coverings of a special type (the so-called tangent and cotangent one) are efficient in analysis and construction of Hamiltonian structures and recursion operators, see [4]. A very interesting development in the theory of coverings can also be found in [5].

In this paper, we solve the following naturally arising problem: let a covering $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ be given and assume that the equation \mathcal{E} is known to possess infinite number of symmetries and/or conservation laws. Is $\tilde{\mathcal{E}}$ endowed with similar properties? The answer, under reasonable assumptions, is positive.

In Section 1, we present a short introduction to the theory of coverings based mainly on [3] and formulate and prove necessary auxiliary facts. Section 2 contains the proof of the main result for the case of Abelian coverings. Finally, the non-Abelian case is discussed in Section 3.

1. Basic notions

For a detailed exposition of the geometrical approach to PDEs we refer the reader to the books [6,7]. Coverings are discussed in [3].

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Equations

Let M be a smooth manifold, $\dim M = n$, and $\pi: E \rightarrow M, \dim E = m + n$, be a locally trivial vector bundle. Consider an infinitely prolonged differential equation $\mathcal{E} \subset J^\infty(\pi)$ embedded to the space of infinite jets. One has the surjection $\pi_\infty: \mathcal{E} \rightarrow M$. The main geometric structure on \mathcal{E} is the *Cartan connection* $\mathcal{C}: Z \mapsto \mathcal{C}_Z$ that takes vector fields on M to those on \mathcal{E} . Vector fields of the form \mathcal{C}_Z are called *Cartan fields*. The connection is flat, i.e., $\mathcal{C}_{[Z,Z']} = [\mathcal{C}_Z, \mathcal{C}_{Z'}]$ for any vector fields on M . The corresponding horizontal distribution (the *Cartan distribution*) on \mathcal{E} is integrable and its maximal integral manifolds are solutions of \mathcal{E} . We always assume \mathcal{E} to be *differentially connected* which means that for any set of linearly independent vector fields Z_1, \dots, Z_n on M the system

$$\mathcal{C}_{Z_i}(h) = 0, \quad i = 1, \dots, n,$$

has constant solutions only.

If x^1, \dots, x^n are local coordinates on M then the Cartan connection takes the partial derivatives $\partial/\partial x^i$ to the *total derivatives* D_{x^i} on \mathcal{E} . Flatness of \mathcal{C} amounts to the fact that the total derivatives pair-wise commute, $[D_{x^i}, D_{x^j}] = 0$.

A π_∞ -vertical vector field S is a *symmetry* of \mathcal{E} if it commutes with all Cartan fields, i.e., $[S, \mathcal{C}_Z] = 0$ for all X . The set of symmetries is a Lie algebra over \mathbb{R} denoted by $\text{sym } \mathcal{E}$.

A differential q -form ω on $\mathcal{E}, q = 0, 1, \dots, n$, is *horizontal* if $i_V \omega = 0$ for any π_∞ -vertical field V . The space of these forms is denoted by $\Lambda_h^q(\mathcal{E})$. Locally, horizontal forms are

$$\omega = \sum a_{i_1, \dots, i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q}, \quad a_{i_1, \dots, i_q} \in \mathcal{F}(\mathcal{E}).$$

The *horizontal de Rham differential* $d_h: \Lambda_h^q(\mathcal{E}) \rightarrow \Lambda_h^{q+1}(\mathcal{E})$ is defined, whose action is locally presented by

$$d_h(a_{i_1, \dots, i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q}) = \sum_{i=1}^n D_{x^i}(a_{i_1, \dots, i_q}) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q}.$$

A closed horizontal $(n - 1)$ -form is called a *conservation law* of \mathcal{E} . Thus, conservation laws are defined by $d_h \omega = 0, \omega \in \Lambda_h^{n-1}(\mathcal{E})$. A conservation law is *trivial* if $\omega = d_h \rho$ for some $\rho \in \Lambda_h^{n-2}(\mathcal{E})$. The quotient space of all conservation laws modulo trivial ones is denoted by $\text{cl } \mathcal{E}$.

If $S \in \text{sym } \mathcal{E}$ and ω is a conservation law then the Lie derivative $L_S \omega$ is a conservation law as well and trivial conservation laws are taken to trivial ones. Thus we have a well-defined action $L_S: \text{cl } \mathcal{E} \rightarrow \text{cl } \mathcal{E}$.

Coverings

Let us now give the main definition. Consider a locally trivial vector bundle $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ of rank r and denote by $\mathcal{F}(\mathcal{E})$ and $\mathcal{F}(\tilde{\mathcal{E}})$ the algebras of smooth functions on \mathcal{E} and $\tilde{\mathcal{E}}$, respectively. We have the embedding $\tau^*: \mathcal{F}(\mathcal{E}) \hookrightarrow \mathcal{F}(\tilde{\mathcal{E}})$.

Definition 1. We say that τ carries a *covering structure* (or is a *differential covering* over \mathcal{E}) if: (a) there exists a flat connection $\tilde{\mathcal{C}}$ in the bundle $\pi_\infty \circ \tau: \tilde{\mathcal{E}} \rightarrow M$ and (b) this connection enjoys the equation

$$\tilde{\mathcal{C}}_Z|_{\mathcal{F}(\mathcal{E})} = \mathcal{C}_Z$$

for all vector fields Z on M .

In local coordinates, any covering is determined by a system of vector fields

$$\tilde{D}_{x^i} = D_{x^i} + X_i, \quad i = 1, \dots, n, \tag{1}$$

on $\tilde{\mathcal{E}}$, where X_i are τ -vertical fields that satisfy the relations

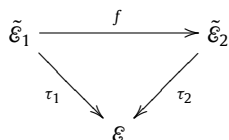
$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \quad 1 \leq i < j \leq n. \tag{2}$$

Let w^1, \dots, w^r be local coordinates in the fiber of τ (the *nonlocal variables* in τ) and $X_i = X_i^1 \partial/\partial w^1 + \dots + X_i^r \partial/\partial w^r$. Then $\tilde{\mathcal{E}}$, endowed with $\tilde{\mathcal{C}}$, is equivalent to the overdetermined system of PDEs

$$\frac{\partial w^\alpha}{\partial x^i} = X_i^\alpha, \quad i = 1, \dots, n, \alpha = 1, \dots, r, \tag{3}$$

compatible by virtue of \mathcal{E} .

Two coverings $\tau_i: \tilde{\mathcal{E}}_i \rightarrow \mathcal{E}, i = 1, 2$, are *equivalent* if there exists a diffeomorphism $f: \tilde{\mathcal{E}}_1 \rightarrow \tilde{\mathcal{E}}_2$ such that the diagram



is commutative and $f_* \circ \tilde{\mathcal{C}}_Z^1 = \tilde{\mathcal{C}}_Z^2$ for all fields Z on M , where $\tilde{\mathcal{C}}^i$ is the Cartan connection on $\tilde{\mathcal{E}}_i$ and f_* is the differential of f .

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