Journal of Geometry and Physics 87 (2015) 296–304

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Integrability in differential coverings

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ARTICLE INFO

Article history:
Available online 3 January 2014

MSC:
37K05
37K10
37K35

Keywords:
Geometry of differential equations
Integrability
Symmetries
Conservation laws
Differential coverings

ABSTRACT

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a differential covering of a PDE $\tilde{\mathcal{E}}$ over $\mathcal{E}$. We prove that if $\mathcal{E}$ possesses infinite number of symmetries and/or conservation laws then $\tilde{\mathcal{E}}$ has similar properties.

0. Introduction

The notion of a covering (or, better, *differential* covering) was introduced by Vinogradov in [1] and elaborated in detail later in [2,3]. Coverings, explicitly or implicitly, provide an adequate background to deal with nonlocal aspects in the geometry of PDEs (nonlocal symmetries and conservation laws, Wahlquist–Estabrook prolongation structures, Lax pairs, zero-curvature representations, etc.). Coverings of a special type (the so-called tangent and cotangent one) are efficient in analysis and construction of Hamiltonian structures and recursion operators, see [4]. A very interesting development in the theory of coverings can also be found in [5].

In this paper, we solve the following naturally arising problem: let a covering $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be given and assume that the equation $\mathcal{E}$ is known to possess infinite number of symmetries and/or conservation laws. Is $\tilde{\mathcal{E}}$ endowed with similar properties? The answer, under reasonable assumptions, is positive.

In Section 1, we present a short introduction to the theory of coverings based mainly on [3] and formulate and prove necessary auxiliary facts. Section 2 contains the proof of the main result for the case of Abelian coverings. Finally, the non-Abelian case is discussed in Section 3.

1. Basic notions

For a detailed exposition of the geometrical approach to PDEs we refer the reader to the books [6,7]. Coverings are discussed in [3].

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0393-0440/$ – see front matter © 2014 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.geomphys.2013.12.009

Equations

Let M be a smooth manifold, $\dim M = n$, and $\pi\colon E \to M$, $\dim E = m + n$, be a locally trivial vector bundle. Consider an infinitely prolonged differential equation $\mathcal{E} \subset J^\infty(\pi)$ embedded to the space of infinite jets. One has the surjection $\pi_\infty\colon \mathcal{E} \to M$. The main geometric structure on $\mathcal{E}$ is the *Cartan connection* $\mathcal{C}\colon Z \mapsto \mathcal{C}_Z$ that takes vector fields on M to those on $\mathcal{E}$. Vector fields of the form $\mathcal{C}_Z$ are called *Cartan fields*. The connection is flat, i.e., $\mathcal{C}_{[Z,Z']} = [\mathcal{C}_Z, \mathcal{C}_{Z'}]$ for any vector fields on M. The corresponding horizontal distribution (the *Cartan distribution*) on $\mathcal{E}$ is integrable and its maximal integral manifolds are solutions of $\mathcal{E}$. We always assume $\mathcal{E}$ to be *differentially connected* which means that for any set of linearly independent vector fields $Z_1, \dots, Z_n$ on M the system

$$\mathcal{C}_{Z_i}(h) = 0, \quad i = 1, \dots, n,$$

has constant solutions only.

If $x^1, \dots, x^n$ are local coordinates on M then the Cartan connection takes the partial derivatives $\partial/\partial x^i$ to the *total derivatives* D_{x^i} on $\mathcal{E}$. Flatness of $\mathcal{C}$ amounts to the fact that the total derivatives pair-wise commute, $[D_{x^i}, D_{x^j}] = 0$.

A π_∞-vertical vector field S is a *symmetry* of $\mathcal{E}$ if it commutes with all Cartan fields, i.e., $[S, \mathcal{C}_Z] = 0$ for all X. The set of symmetries is a Lie algebra over $\mathbb{R}$ denoted by $\operatorname{sym}\mathcal{E}$.

A differential q-form ω on $\mathcal{E}$, $q = 0, 1, \dots, n$, is *horizontal* if $\mathrm{i}_V\omega = 0$ for any π_∞-vertical field V. The space of these forms is denoted by $\Lambda_h^q(\mathcal{E})$. Locally, horizontal forms are

$$\omega = \sum a_{i_1,\dots,i_q} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_q}, \quad a_{i_1,\dots,i_q} \in \mathcal{F}(\mathcal{E}).$$

The *horizontal de Rham differential* $\mathrm{d}_h\colon \Lambda_h^q(\mathcal{E}) \to \Lambda_h^{q+1}(\mathcal{E})$ is defined, whose action is locally presented by

$$\mathrm{d}_h(a_{i_1,\dots,i_q} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_q}) = \sum_{i=1}^n D_{x^i}(a_{i_1,\dots,i_q}) \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_q}.$$

A closed horizontal $(n-1)$-form is called a *conservation law* of $\mathcal{E}$. Thus, conservation laws are defined by $\mathrm{d}_h\omega = 0$, $\omega \in \Lambda_h^{n-1}(\mathcal{E})$. A conservation law is *trivial* if $\omega = \mathrm{d}_h\rho$ for some $\rho \in \Lambda_h^{n-2}(\mathcal{E})$. The quotient space of all conservation laws modulo trivial ones is denoted by $\operatorname{cl}\mathcal{E}$.

If $S \in \operatorname{sym}\mathcal{E}$ and ω is a conservation law then the Lie derivative $\mathrm{L}_S\omega$ is a conservation law as well and trivial conservation laws are taken to trivial ones. Thus we have a well-defined action $\mathrm{L}_S\colon \operatorname{cl}\mathcal{E} \to \operatorname{cl}\mathcal{E}$.

Coverings

Let us now give the main definition. Consider a locally trivial vector bundle $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ of rank r and denote by $\mathcal{F}(\mathcal{E})$ and $\mathcal{F}(\tilde{\mathcal{E}})$ the algebras of smooth functions on $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively. We have the embedding $\tau^*\colon \mathcal{F}(\mathcal{E}) \hookrightarrow \mathcal{F}(\tilde{\mathcal{E}})$.

Definition 1. We say that τ carries a *covering structure* (or is a *differential covering* over $\mathcal{E}$) if: (a) there exists a flat connection $\tilde{\mathcal{C}}$ in the bundle $\pi_\infty \circ \tau\colon \tilde{\mathcal{E}} \to M$ and (b) this connection enjoys the equation

$$\tilde{\mathcal{C}}_Z\big|_{\mathcal{F}(\mathcal{E})} = \mathcal{C}_Z$$

for all vector fields Z on M.

In local coordinates, any covering is determined by a system of vector fields

$$\tilde{D}_{x^i} = D_{x^i} + X_i, \quad i = 1, \dots, n, \tag{1}$$

on $\tilde{\mathcal{E}}$, where X_i are τ-vertical fields that satisfy the relations

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \quad 1 \le i < j \le n. \tag{2}$$

Let $w^1, \dots, w^r$ be local coordinates in the fiber of τ (the *nonlocal variables* in τ) and $X_i = X_i^1\partial/\partial w^1 + \dots + X_i^r\partial/\partial w^r$. Then $\tilde{\mathcal{E}}$, endowed with $\tilde{\mathcal{C}}$, is equivalent to the overdetermined system of PDEs

$$\frac{\partial w^\alpha}{\partial x^i} = X_i^\alpha, \quad i = 1, \dots, n, \ \alpha = 1, \dots, r, \tag{3}$$

compatible by virtue of $\mathcal{E}$.

Two coverings $\tau_i\colon \tilde{\mathcal{E}}_i \to \mathcal{E}$, $i = 1, 2$, are *equivalent* if there exists a diffeomorphism $f\colon \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{E}}_1 & \xrightarrow{\quad f \quad} & \tilde{\mathcal{E}}_2 \\ & {\scriptstyle\tau_1}\searrow \quad \swarrow{\scriptstyle\tau_2} & \\ & \mathcal{E} & \end{array}$$

is commutative and $f_* \circ \tilde{\mathcal{C}}_Z^1 = \tilde{\mathcal{C}}_Z^2$ for all fields Z on M, where $\tilde{\mathcal{C}}^i$ is the Cartan connection on $\tilde{\mathcal{E}}_i$ and f_* is the differential of f.