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Induced representations and traces for chains of affine and cyclotomic Hecke algebras

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ABSTRACT

Properties of relative traces and symmetrizing forms on chains of cyclotomic and affine Hecke algebras are studied. The study relies on the use of bases of these algebras which generalize a normal form for elements of the complex reflection groups $G(m, 1, n)$, $m = 1, 2, \ldots, \infty$, constructed by a recursive use of the Coxeter–Todd algorithm. Formulas for inducing, from representations of an algebra in the chain, representations of the next member of the chain are presented.

1. Introduction

The Coxeter–Todd algorithm [1] is a powerful tool for constructing a normal form for group elements with respect to a given subgroup. For an ascending chain of groups the Coxeter–Todd algorithm establishes recursively a global normal form for group elements. We apply the algorithm to the chain, in n, of the complex reflection groups $G(m, 1, n)$. The algorithm works uniformly for all $n > 1$. The normal form of an element of $G(m, 1, n)$ is $\text{L}w$, where $w \in G(m, 1, n-1)$ and L runs through a left transversal of the subgroup $G(m, 1, n-1)$ in the group $G(m, 1, n)$; moreover, the left transversal is uniform as well. In this sense (that the normal form is $\text{L}w$), this normal form is "inductive", in contrast to the normal forms used in [2,3]; as in [3], this normal form consists of reduced expressions in terms of generators of $G(m, 1, n)$.

We allow the value $m = \infty$. The group $G(\infty, 1, n)$ is the affine Weyl group of type GL. The Coxeter–Todd algorithm is applicable for $m = \infty$ and we find the corresponding left transversal and the normal form for group elements.

We denote by $H(m, 1, n)$ the standard deformation of the complex reflection group $G(m, 1, n)$; it has been introduced in [2,4,5] and is called the Ariki–Koike algebra, or the cyclotomic Hecke algebra. Again, $m = \infty$ is allowed, it corresponds to the affine Hecke algebra $\hat{H}_n$ of type GL. The above inductive normal form has a nice generalization to a basis $\mathcal{B}$ of the algebra

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$H(m, 1, n)$, $m = 1, 2, \ldots, \infty$, inductive with respect to the chain, in n. The use of the basis $\mathcal{B}$ allows for a representation-free approach, avoiding a dimension count, to the structure theory of the cyclotomic and affine Hecke algebras. The fact, that $\mathcal{B}$ is a basis, implies that $H(m, 1, n)$ is a flat deformation of $\mathbb{C}G(m, 1, n)$. The basis $\mathcal{B}$ is different from the bases in [2,3] (introduced there for $m < \infty$) but is similar to the basis in [6] used for the study of Markov traces on the (cyclotomic) Hecke algebra. Nevertheless the results of [6] rely on the basis in [2], obtained as an outcome of the classification of irreducible representations of the algebra $H(m, 1, n)$, for $m < \infty$. Our arguments do not refer to the representation theory and work for $m = \infty$ as well; the proof is done more in a spirit of classical proofs for the usual Hecke algebra (note that a basis of cyclotomic quotients of the degenerate affine Hecke algebras is established in [7] without the use of the representation theory).

A key ingredient in our study consists in explicit formulas for inducing representations of the algebra $H(m, 1, n-1)$ to representations of the algebra $H(m, 1, n)$, $m = 1, 2, \ldots, \infty$. A one-dimensional representation of the algebra $H(m, 1, n-1)$ induces a natural analogue, for the algebra $H(m, 1, n)$, of the Burau representation.

The cyclotomic/affine Hecke algebras possess commutative Bethe subalgebras (see [8] for the symmetric group case). The Bethe subalgebras play a fundamental role in the theory of integrable systems such as chain and Gaudin models, see, e.g., [9,10]. One of the ways to construct the Bethe subalgebras is based on "relative traces", the linear maps $\mathrm{Tr}_k\colon H(m, 1, k) \to H(m, 1, k-1)$, satisfying certain conditions, see Section 5 for precise definitions. We use the basis $\mathcal{B}$ to establish the existence and uniqueness properties of the relative traces for the chain of the cyclotomic/affine Hecke algebras.

A composition $\mathrm{Tr} := \mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_{n-1} \circ \mathrm{Tr}_n$ of relative traces defines a Markov trace on the algebra $H(m, 1, n)$ and all Markov traces constructed in [6] can be thus obtained. In particular, the existence and unicity of the Markov traces constructed in [6] is reobtained in a different way via the study of the relative traces. If Tr is non-degenerate on both $H(m, 1, n)$ and $H(m, 1, n-1)$ then Tr_n coincides with the conditional expectation.

Besides, there is a family L^γ of central forms on $H(m, 1, n)$ which possess a multiplicativity property with respect to the basis $\mathcal{B}$. Up to the standard involution of $H(m, 1, n)$, these central forms L^γ are Markov traces on $H(m, 1, n)$ with the "Markov parameter" equal to 0 (see Section 6 for precisions). As an application of use of the basis $\mathcal{B}$, we obtain expressions for the products of the generators and the basis elements and give an independent direct proof of the centrality of the linear forms L^γ on $H(m, 1, n)$. For $m < \infty$, one of these central forms, denoted L^{γ°, coincides with the central form introduced in [3] and further studied in [11]. In the terminology of [11], the inductive basis $\mathcal{B}$ is *quasi-symmetric* with respect to the central form L^{γ°, as are the different bases in [2,3]. We also note that, for each central form L^γ, an easy modification of the inductive basis $\mathcal{B}$ yields a basis quasi-symmetric with respect to L^γ.

In [12], a *fusion formula* for a complete set of primitive idempotents of the algebra $H(m, 1, n)$, for finite m, is obtained. The fusion formula is well adapted to the inductive basis $\mathcal{B}$ (for more details, see also [13]). For $m < \infty$, it turns out that, due to the multiplicativity property, the weights of the central forms L^γ can be easily calculated with the help of the fusion formula for the algebra $H(m, 1, n)$. The weights of any Markov trace from [6] have been calculated in [14] (the weights of the form L^{γ° have been calculated independently in [15], see also [16]). The formulas for the weights obtained here are different from the formulas in [14] and generalize, for any L^γ, the so-called "cancellation-free" formula obtained in [16] for the weights of L^{γ°.

The paper is organized as follows.

In Section 2, we present the Coxeter–Todd algorithm for the chain (with respect to n) of the groups $G(m, 1, n)$. We establish the resulting normal form for elements of $G(m, 1, n)$.

The normal form for elements of $G(m, 1, n)$ suggests a basis for the algebra $H(m, 1, n)$. In Sections 3 and 4, we show that this is indeed a basis. Several known facts about the chain (with respect to n) of the algebras $H(m, 1, n)$ are reestablished with the help of this basis. In particular, we show that $H(m, 1, n)$ is a flat deformation of the group ring of $G(m, 1, n)$. We also give the formulas for the induced representations.

Section 5 is devoted to the relative traces for the chain of the cyclotomic/affine Hecke algebras.

In Section 6, we complete the study of the multiplication of the generators on the basis elements. We then use the results to establish properties of central forms L^γ on $H(m, 1, n)$. For the cyclotomic Hecke algebras, we present a calculation, based on the fusion formula, of weights of the forms L^γ in Section 6.3.

Notation. $\mathfrak{C}_m := \{0, 1, \ldots, m-1\}$ for finite m and $\mathfrak{C}_\infty := \mathbb{Z}$;
$\mathcal{A}_m := \mathbb{C}[q^{\pm 1}, v_1^{\pm 1}, \ldots, v_m^{\pm 1}]$ for finite m and $\mathcal{A}_\infty := \mathbb{C}[q, q^{-1}]$; here $q, v_1, \ldots, v_m$ are indeterminates.
The symbol □ stands for an end of a proof, △ – for an end of a remark.

2. Normal form for the group $G(m, 1, n)$

Let $m \in \mathbb{Z}_{>0} \cup \{\infty\}$. The group $G(m, 1, n)$ is generated by the elements $t, s_1, \ldots, s_{n-1}$ with the defining relations:

$$\begin{cases} s_i^2 = 1 & \text{for } i = 1, \ldots, n-1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i = 1, \ldots, n-2, \\ s_i s_j = s_j s_i & \text{for all } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1, \end{cases} \tag{1}$$

and

$$\begin{cases} t^m = 1 & \text{if } m < \infty, \\ t s_1 t s_1 = s_1 t s_1 t, & \\ t s_i = s_i t & \text{for } i = 2, \ldots, n-1. \end{cases} \tag{2}$$