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Open problems in the theory of finite-dimensional integrable systems and related fields

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ABSTRACT

This paper collects a number of open problems in the theory of integrable systems and related fields, their study being suggested by the main lecturers and participants of the *Advanced Course on Geometry and Dynamics of Integrable Systems*, from September 9th to 14th 2013, as well as the *Conference on Integrability, Topological Obstructions to Integrability and Interplay with Geometry*, from September 16th to 20th 2013, both held at the Centre de Recerca Mathematica in Barcelona within the Research Programme "Geometry and Dynamics of Integrable Systems".

1. Introduction

The term "integrability" in geometry and mathematical physics is fairly overloaded. For a dynamical system, it means in its broadest sense the existence of a certain, in general singular, foliation in the phase space by leaves which are invariant under the flow of the system. For Hamiltonian systems, this translates to the existence of a family of *first integrals* – i.e. functions that are constant along the trajectories of the system – which commute pairwise under the Poisson bracket.

Invariant foliations in phase space define restrictions on the possible evolution of the system. We have thus not only *integrability* versus *non-integrability*, but a finite scale of integrability, measured by the dimension of the leaves or, when dealing with a Hamiltonian system, the dimension of a space of commuting first integrals. If the number of first integrals is maximal, the system is said to be completely integrable or *integrable in the sense of Liouville*.

But why are we looking for integrable dynamical systems? The ultimate goal is to understand the dynamics of the system. In the strongest sense this means to solve – or "integrate", as one often says – the equations of motion explicitly. But the meaning of "explicitly" is a rather philosophical question, as most of the "explicit" solutions are defined as solutions of their defining differential equations. If we want to avoid such tautological solutions, we have to restrict the possible operations we allow when we *integrate* an ordinary differential equation. This leads to *integrability by quadratures*.

Let us briefly recall the notion of integrability in its most frequently used sense: that of Liouville integrability. We also take this opportunity to introduce common notations which will be used throughout the entire article.

Let (M, ω) be a symplectic manifold and let X_f be the Hamiltonian vector field corresponding to a function $f : M \to \mathbb{R}$, defined by the formula $i_{X_f}\omega = -df$. The canonical Poisson bracket between functions f, g is given by $\{f, g\} = \omega(X_f, X_g)$. A Liouville integrable system is a collection of $m = \frac{1}{2}\dim M$ functions $F_1, F_2, \ldots, F_m$ which mutually Poisson commute and are

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functionally independent almost everywhere. The Arnold–Liouville theorem [1] then assures the existence of a canonical transformation such that the transformed Hamiltonian system can be integrated explicitly in terms of quadratures, at least on an open dense subset. In mechanics and applications, one is mostly interested in a particular dynamical system given by the flow of a vector field X_H, where the Hamiltonian functions H describes a particular mechanical setting. Such a system is Liouville integrable, if there exist functions $F_2, \dots, F_m$ which together with $F_1 = H$ constitute a Liouville integrable systems as described above.

The Research Programme "Geometry and Dynamics of Integrable Systems" [2], held at the Centre de Recerca Mathemàtica (CRM) in Barcelona, has brought together several experts working on different aspects of integrable systems. The following is a collection of open problems proposed by them during the *Advanced Course on Geometry and Dynamics of Integrable Systems*, from September 9th to 14th 2013, and the subsequent *Conference on Integrability, Topological Obstructions to Integrability and Interplay with Geometry*, from September 16th to 20th 2013.

The spectrum of problems presented here reflects the diversity of the Theory of Integrable Systems and the various forms of integrable systems: In Section 2, Liouville integrable systems coming from two compatible Poisson structures are considered. Section 3 is concerned with the extension of the definition of integrability to non-Hamiltonian systems. In Section 4, integrable systems are regarded from the point of view of their induced actions. The notion of Liouville integrability for ordinary differential equations and its relation to classical Liouville integrability is at the center of Sections 5 and 6 deals with the relations between integrability and dynamical complexity. Finally, in the last section, integrability is regarded for certain non-holonomic systems.

Many other notions of integrability have been proposed, for example discrete, infinite dimensional or non-commutative integrable systems, which are close to the present setting but shall not be discussed here.

2. Bi-Hamiltonian systems

Since the pioneering work of Magri [3], the theory of bi-Hamiltonian systems and compatible Poisson brackets has gained much interest. It has been observed that the integrability in many classical Hamiltonian systems has a bi-Hamiltonian nature and, on the other hand, the bi-Hamiltonian approach led to the construction of new examples of integrable Hamiltonian systems, see the articles [4,5] and the references therein.

In Section 2.1, we will describe a purely linear-algebraic approach to compatible Poisson brackets that studies the situation obtained by restricting the brackets to a point of the manifold. This approach already helps to understand the basic principles underlying the theory of bi-Hamiltonian systems. Our consideration in this section follows closely the content of [6,4,7]. In Section 2.2, we explain the open problems that appeared in this setting.

We will start with recalling some general facts on bi-Hamiltonian systems and compatible Poisson brackets. A Poisson structure $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on a manifold M can be written in the form $\{f, g\} = A(df, dg)$, where $A \in \Gamma(\Lambda^2 TM)$ is a skew-symmetric tensor field of type $(2, 0)$ called the Poisson tensor. The Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to a nonlinear first order PDE system on the components of the tensor A. In case that the Poisson structure is the canonical one corresponding to a symplectic structure, the Poisson tensor A is just given by the dual of the symplectic form but in general, the natural mapping $A : T^*M \to TM$ (given in local coordinates $x_1, \dots, x_n$ by $\alpha = \alpha_i dx^i \longmapsto A^{ij}\alpha_i\partial_j$) is degenerate. Generalizing the definition from the symplectic setting, we can define the Hamiltonian vector field corresponding to a function f on M by $X_f = A(df)$.

An integrable system on a Poisson manifold M is a complete family of functions $\mathcal{F}$ in involution. Involutivity means that the functions in $\mathcal{F}$ commute with each other whilst completeness means that $\mathcal{F}$ contains $\frac{1}{2}(\dim M + \operatorname{corank} A)$ independent functions, where $\operatorname{corank} A = \dim M - \operatorname{rank} A$ and the rank of A is defined by $\operatorname{rank} A = \max\{\operatorname{rank}(A_x : T_x^*M \to T_xM) : x \in M\}$.

Two Poisson structures $\{.,.\}_A$, $\{.,.\}_B$ with corresponding Poisson tensors A, B are called compatible if their sum $\{f, g\} = \{f, g\}_A + \{f, g\}_B$ (and hence any linear combination) is again a Poisson structure. The only obstruction for compatibility is thus that the sum satisfies the Jacobi identity. We now come to a very important example, that of compatible Lie–Poisson structures, which also plays a crucial role in our open problems (see Sections 2.2.3 and 2.2.4).

Example 1 (*Compatible Lie–Poisson Structures*)**.** Let $\mathfrak{g}^*$ be the dual of a finite-dimensional Lie algebra $\mathfrak{g}$ with Lie bracket $[.,.]$. The Poisson structure defined by

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle,$$

for $f, g \in C^\infty(\mathfrak{g}^*)$, $x \in \mathfrak{g}^*$, is called the *standard Lie–Poisson structure*. Here $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ and $df(x), dg(x) : \mathfrak{g}^* \to \mathbb{R}$ are considered as elements of $\mathfrak{g}$. The value $A(x)$ of the corresponding Poisson tensor A at a point $x \in \mathfrak{g}^*$ is a 2-form on $T_x^*\mathfrak{g}^* \cong \mathfrak{g}$ given by $A(x)(\xi, \eta) = \langle x, [\xi, \eta]\rangle$.

For fixed $a \in \mathfrak{g}^*$, we define the *constant Lie–Poisson bracket*

$$\{f, g\}_a(x) = \langle a, [df(x), dg(x)]\rangle$$

with constant Poisson tensor A_a whose value at x is given by the constant tensor $A_a(x)(\xi, \eta) = \langle a, [\xi, \eta]\rangle$. This bracket is compatible with the standard Lie–Poisson bracket from above.