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Graded geometry in gauge theories and beyond

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ABSTRACT

We study some graded geometric constructions appearing naturally in the context of gauge theories. Inspired by a known relation of gauging with equivariant cohomology we generalize the latter notion to the case of arbitrary Q-manifolds introducing thus the concept of equivariant Q-cohomology. Using this concept we describe a procedure for analysis of gauge symmetries of given functionals as well as for constructing functionals (sigma models) invariant under an action of some gauge group.

As the main example of application of these constructions we consider the twisted Poisson sigma model. We obtain it by a gauging-type procedure of the action of an essentially infinite dimensional group and describe its symmetries in terms of classical differential geometry.

We comment on other possible applications of the described concept including the analysis of supersymmetric gauge theories and higher structures.

1. Introduction/preliminaries

In this paper we describe the possibilities offered by graded geometry for the analysis of physical theories and some objects of classical differential geometry. We introduce a powerful tool—the concept of equivariant Q-cohomology which is a natural extension of the definition of standard equivariant cohomology to Q-manifolds.

In the first part of the paper after introducing the problem of gauging we briefly sketch some facts from the theory of Q-manifolds and fix the notations in the examples that are important in what follows. In Section 2 we define the notion of equivariant Q-cohomology and explain, following the scheme of A. Kotov and T. Strobl [1,2], its relation to gauge invariance. Within this framework we recover explicitly the result of J.M. Figueroa-O'Farrill and S. Stanciu [3] on the obstruction to gauging of the Wess–Zumino terms—this shows also how ordinary equivariant cohomology can be obtained as a particular case of Q-cohomology. Section 3 is entirely devoted to the analysis of the twisted Poisson sigma model [4]: we describe the algebra of its symmetries and construct its functional using the procedure that we suggest as an alternative to standard gauging. In Section 4 we give a purely mathematical application of the described concept, namely we propose a possible definition of equivariant cohomology for Courant algebroids. To conclude we also comment on other applications and some work in progress.

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1.1. Gauging problem, Wess–Zumino terms

The major part of this work is motivated by the gauging problem, which is important in theoretical physics. To give a simple example of the procedure consider $X\colon \Sigma^d \to M^n$—a map between two smooth manifolds of dimensions d and n respectively, and $B \in \Omega^d(M)$; in the physicist's terminology one would call X a scalar field, Σ—worldsheet and M—target. Assume that a Lie group G acts on M and leaves B invariant. This induces a G-action on M^Σ, which leaves invariant the functional $S[X] = \int_\Sigma X^*B$. The invariance with respect to G is called a (global) *rigid invariance*. The functional is called (locally) *gauge invariant*, if it is invariant with respect to the group $G^\Sigma \equiv C^\infty(\Sigma, G)$; it is clear that in its original form $S[X]$ is not necessarily gauge invariant.

The procedure of gauging consists in modifying the functional S in order to make it gauge invariant. This is usually done by introducing new variables to S controlling however that the result reduces to the initial functional when the additional variables are put to zero—these new variables may have some physical meaning in concrete applications. For the above functional the gauging problem can be solved by extending S to a functional $\widetilde{S}$ defined on $(X, A) \in M^\Sigma \times \Omega^1(\Sigma, \mathfrak{g})$, $\mathfrak{g} = Lie(G)$ by means of so-called minimal coupling. For example if $d = 2$ the result is

$$\widetilde{S}[X, A] = \int_\Sigma \left(X^*B - A^a X^* \iota_{v_a} B + \frac{1}{2} A^a A^b X^* \iota_{v_a} \iota_{v_b} B \right),$$

where A^a, A^b, ι_{v_a}, ι_{v_b} are defined by fixing the basis of $\mathfrak{g}$.[1] Making again a remark about the physicist's terminology one would call the variables A the one-form valued (gauge) fields.

Suppose now that $\Sigma^d = \partial \Sigma^{d+1}$, $\tilde{X}\colon \Sigma^{d+1} \to M$—an extension of X coinciding with it on Σ^d. Let $H \in \Omega^{d+1}(M)$ be a closed differential form invariant under the induced action of G. The functional $S[X] = \int_{\Sigma^{d+1}} \tilde{X}^*H$ is the simplest one containing the so-called *Wess–Zumino term* (the integration is performed over the bulk of the worldsheet manifold, [5]). In contrast to the previous example gauging of this functional can be obstructed. More precisely in [3,6] it has been shown that gauging is possible, if and only if H permits an equivariantly closed extension. There is however a serious limitation for the procedure suggested in [3], namely the number of introduced gauge fields is equal to the dimension of the group G making it not very practical for essentially infinite dimensional groups which do appear in applications. In what follows we will see how this issue can be treated within the framework of Q-bundles and in particular observe the situations when the extension of the target is not given explicitly by the Lie algebra of the group acting.

1.2. Graded geometry, Q-manifolds

We are certainly not going to give a full introduction to graded geometry, referring to nice sources like [7–10]. Let us however give a definition of a Q-manifold and several examples of it.

Definition 1.1. A *Q-manifold* is graded manifold equipped with a *Q-structure*—a degree 1 vector field Q satisfying $[Q, Q] \equiv 2Q^2 = 0$.

Example 1.1. Consider any smooth manifold M, declare fiber linear coordinates of the tangent bundle to it to be of degree 1. The graded manifold obtained like this is generally denoted by $T[1]M$. It is equipped with the de Rham differential that written in local coordinates $(x^i, \theta^i = dx^i)$ has the form $d_{dR} = \theta^i \frac{\partial}{\partial x^i}$, it thus can be viewed as a degree 1 vector field squaring to zero.

Example 1.2. Consider a Lie algebra $\mathfrak{g}$, choose a basis of it and declare local coordinates ξ^a to be of degree 1. This graded manifold denoted $\mathfrak{g}[1]$ is equipped with the Chevalley–Eilenberg differential $Q_{CE} = C^a_{bc} \xi^b \xi^c \frac{\partial}{\partial \xi^a}$, where C^a_{bc} are the structure constants of $\mathfrak{g}$. $Q_{CE}^2 = 0$ is equivalent to the Jacobi identity.

A more involved example is provided by a twisted Poisson manifold M. Let us recall that given a closed differential form $H \in \Omega^3(M)$ an almost Poisson bivector Π is called *twisted Poisson* if and only if it satisfies the twisted version of the Jacobi identity: $[\Pi, \Pi]_{SN} = (\Pi^\#)^{\otimes 3}(H)$, where $[\cdot; \cdot]_{SN}$ is the Schouten–Nijenhuis bracket of multivector fields and the right hand side of the equality denotes the full contraction of H with Π. In this case the couple (Π, H) is called a *twisted Poisson structure*.

Example 1.3. Consider a cotangent bundle to a manifold M equipped with a twisted Poisson structure (Π, H). A graded manifold obtained by shifting the grading of the fiber linear coordinates p_i by 1 is usually denoted by $T^*[1]M$. For $C^{jk}_i(x) = \frac{\partial \pi^{jk}}{\partial x^i} + H_{ij'k'} \pi^{jj'} \pi^{kk'}$ consider the degree 1 vector field

$$Q_{\pi,H} = \pi^{ij} p_j \frac{\partial}{\partial x^i} - \frac{1}{2} C^{jk}_i p_j p_k \frac{\partial}{\partial p_i}.$$

[1] Here and in the whole text the convention of summation over repeating indices is adopted.