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The pentagram integrals for Poncelet families

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ABSTRACT

The pentagram map is now known to be a discrete integrable system. We show that the integrals for the pentagram map are constant along Poncelet families. That is, if P_1 and P_2 are two polygons in the same Poncelet family, and f is a monodromy invariant for the pentagram map, then $f(P_1) = f(P_2)$. Our proof combines complex analysis with an analysis of the geometry of a degenerating sequence of Poncelet polygons.

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1. Introduction

The *pentagram map* is a projectively natural map defined on the space of n-gons. The case $n = 5$ is classical; it goes back at least to Clebsch in the 19th century and perhaps even to Gauss. Motzkin [1] also considered this case in 1945. I introduced the general version of the pentagram map in 1991. See [2]. I subsequently published two additional papers, [3,4], on the topic. Now there is a growing literature. See the discussion below.

To define the pentagram map, one starts with a polygon P and produces a new polygon $T(P)$, as shown at left in Fig. 1.1 for a convex hexagon. As indicated at right, the map $P \to T^2(P)$ acts naturally on labeled polygons.

The pentagram map is defined on polygons over any field. More generally, as I will discuss below, the pentagram map is defined on the so-called twisted polygons. The pentagram map commutes with projective transformations and thereby induces a map on spaces of projective equivalence classes of polygons, both ordinary and twisted.

In recent years, the pentagram map has attracted a lot of attention, thanks to the following developments.

1. In [4], I found a hierarchy of integrals to the pentagram map, similar to the KdV hierarchy. I also related the pentagram map to the octahedral recurrence, and observed that the continuous limit of the pentagram map is the classical Boussinesq equation. For later reference, call the pentagram integrals the *monodromy invariants*.
2. In [5], Ovsienko, Tabachnikov and I showed that the pentagram map is a completely integrable system when defined on the space of projective classes of twisted polygons. We also elaborated on the connection to the Boussinesq equation. The main new idea is the introduction of a pentagram-invariant Poisson bracket with respect to which the monodromy invariants commute. See also [6].
3. In [7] Soloviev showed that the pentagram map is completely integrable, in the algebro-geometric sense, on spaces of projective classes of real polygons and on spaces of projective classes of complex polygons. In particular Soloviev showed that the pentagram map has a Lax pair and he deduced the invariant Poisson structure from the Phong–Krichever universal formula.

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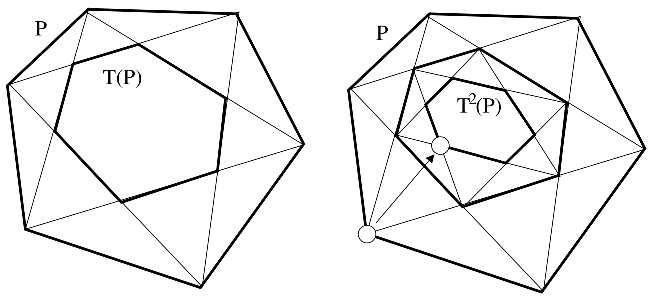

Fig. 1.1. The pentagram map.

4. In [8] (independently, at roughly the same time as [7]) Ovsienko, Tabachnikov and I showed that the pentagram map is a discrete, completely integrable system, in the sense of Liouville–Arnold, when defined on the space of projective classes of closed convex polygons.
5. In [9], Glick identified the pentagram map with a specific cluster algebra, and found algebraic formulas for iterates of the map which are similar in spirit to those found by Robbins and Rumsey for the octahedral recurrence. See also [10].
6. In [11], Gekhtman, Shapiro, Tabachnikov, Vainshtein generalized the pentagram map to similar maps using longer diagonals, and defined on spaces of so-called *corrugated polygons* in higher dimensions. The work in [11] generalizes Glick's cluster algebra.
7. In [12], Mari-Beffa defines higher dimensional generalizations of the pentagram map and relates their continuous limits to various families of integrable PDEs. See also [13].
8. In recent work, [14–16], Khesin and Soloviev obtain definitive results about higher dimensional analogues of the pentagram map, their integrability, and their connection to KdV-type equations.
9. In the preprint [17], Fock and Marshakov relate the pentagram map to, among other things, Poisson Lie groups.
10. The preprint [18] discusses many aspects of the octahedral recurrence, drawing connections to the work in [11].

Though this is not directly related to the pentagram map, it seems also worth mentioning the recent paper [19] of Goncharov and Kenyon, who study a family of cluster integrable systems. These systems are closely related to the octahedral recurrence which, in turn, is closely related to the pentagram map.

A *Poncelet polygon* is a polygon which is simultaneously inscribed in, and circumscribed about, a conic section. Two Poncelet polygons are *in the same family*, or *related*, if they are simultaneously inscribed in the same conic and circumscribed about the same conic. The famous *Poncelet porism* says that any Poncelet polygon is related to a 1-parameter family of Poncelet polygons. What is remarkable here is that the related polygons typically are not projectively equivalent.

The pentagram map interacts nicely with Poncelet polygons. Recall that $T^2(P)$ is the image of P under the square of the pentagram map, considered as a labeled polygon in a canonical way. The following theorem is a consequence of the results in [20], and also a consequence of a classical result of Darboux:

Theorem 1.1. *Let $P, Q \subset \boldsymbol{C}$ be related Poncelet polygons. Then there is a projective transformation (the same for P and Q) which carries $T^2(P)$ to P and $T^2(Q)$ to Q and respects the labelings.*

Note that Theorem 1.1 makes two statements. First, the image of a Poncelet polygon under the square of the pentagram map is projectively equivalent to the original polygon. Second, one and the same projective equivalence works for a pair of related Poncelet polygons.

Let $\mathcal{C}_n$ denote the space of labeled projective equivalence classes of strictly convex real n-gons. A *Poncelet point* in $\mathcal{C}_n$ is an equivalence class of Poncelet polygons. Theorem 1.1 shows that T^2 fixes every Poncelet point in $\mathcal{C}_n$. Every pentagon is a Poncelet polygon, and in fact there is a suitable labeling convention with respect to which T is the identity on $\mathcal{C}_5$. This classical fact was known to Motzkin [1] and perhaps goes back even further. On $\mathcal{C}_6$, the map T^2 is the identity with respect to a labeling convention that is different than the one discussed above: $T^2(P)$ and $\widetilde{P}$ are projectively equivalent, where $\widetilde{P}$ is obtained from P by cycling the vertex labels by 3. For $n \geq 7$, the action of T^2 on $\mathcal{C}_n$ is not periodic.

The purpose of this paper is to study a deeper and more subtle connection between the pentagram map and Poncelet polygons.

Theorem 1.2 (*Main*)**.** *Any two related Poncelet polygons have the same monodromy invariants.*

For convenience, we will prove Theorem 1.2 when n is even and large, say $n > 10$. The odd case has a proof similar to the even case. The case for small n, either even or odd, is similar to the case for large n, but the argument is somewhat less transparent.

Our proof is an argument in complex analysis. When defined over $\boldsymbol{C}$, the generic Poncelet family – i.e., a collection of mutually related Poncelet polygons – is naturally parametrized by a complex torus[1] $\boldsymbol{T}$. The monodromy functions are

1 By *complex torus*, we mean a compact Riemann surface which is holomorphically equivalent to $\boldsymbol{C}/\Lambda$, where Λ a planar lattice.