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Circle actions in geometric quantisation

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ABSTRACT

The aim of this article is to present unifying proofs for results in geometric quantisation with real polarisations by exploring the existence of symplectic circle actions. It provides an extension of Rawnsley's results on the Kostant complex, and gives a partial result for the focus–focus contribution to geometric quantisation; as well as, an alternative proof for theorems of Śniatycki and Hamilton.

1. Introduction

Geometric quantisation tries to associate a Hilbert space to a symplectic manifold via a complex line bundle. Although it is possible to describe the canonical quantisation using this language, most of the difficulties arise when one tries to mimic this procedure for symplectic manifolds which are not naturally cotangent bundles. Those appear in the context of reduction and are far from being artificial mathematical models.

The first difficulty is to isolate in a global way position and momentum, in order to define wave functions from sections of a complex line bundle over the symplectic manifold. This is done by introducing polarisations, which, roughly speaking, are lagrangian foliations. The second issue, that will not be addressed here, is how to define a Hilbert structure; however, all examples treated in this article have a natural one.

Usually, the quantum phase space is constructed using global sections of the line bundle which are flat along the polarisation. In case these global sections do not exist, Kostant suggested to associate quantum states to elements of higher cohomology groups, and to build the quantum phase space from these groups: by considering cohomology with coefficients in the sheaf of flat sections.

At least two approaches can be used to compute these cohomology groups: Čech and de Rham. The results of Hamilton [1] and Hamilton and Miranda [2] are based on a Čech approach, this article takes the de Rham point of view, by finding a resolution for the sheaf. Following Kostant [3,4], a resolution for the sheaf of flat sections can be obtained by twisting the sheaves relative to the foliated complex induced by the polarisation with the sheaf of flat sections.

This article follows closely Rawnsley's ideas [4] and explores the existence of circle actions, in the particular case of real polarisations. The tools developed here highlight and unravel the role played by symplectic circle actions in known results in geometric quantisation, casting some light on a conjecture about the contributions coming from focus–focus type of singularities and providing a different proof for the theorems of Śniatycki [3] and Hamilton [1].

The rest of the article is organised as follows. Section 2 introduces the basic definitions of geometric quantisation. Section 3 summarises relevant results about a resolution of the sheaf of flat sections. Section 4 explores the existence of circle actions: it further develops results from [4] and it contains the main tools of this article. Section 5 presents the important

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notion of Bohr–Sommerfeld leaves. Finally, in Section 6 the tools developed in Section 4 are used to compute the geometric quantisation of local and semilocal models near a focus–focus singularity and fibre, and to prove Śniatycki [3] and Hamilton's [1] theorems concerning geometric quantisation of real polarisations.

Throughout this article and otherwise stated, all the objects considered will be C^∞; manifolds are real, Hausdorff, paracompact, and connected; $C^\infty(V)$ denotes the set of complex-valued functions over V; and the units are such that $\hbar = 1$.

2. Geometric quantisation à la Kostant

An integrable system on a symplectic manifold (M, ω) of dimension $2n$ is a set of n real-valued functions, $f_1, \ldots, f_n \in C^\infty(M)$, satisfying $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$ over an open dense subset of M and $\{f_j, f_k\}_\omega = 0$ for all j, k. The mapping $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ is called a moment map.

The Poisson bracket is defined by $\{f, g\}_\omega = X_f(g)$, where X_f is the unique vector field defined by the equation $\imath_{X_f}\omega = -\mathrm{d}f$, called the hamiltonian vector field of f.

The distribution generated by the hamiltonian vector fields of the moment map is involutive because $[X_f, X_g] = X_{\{f,g\}_\omega}$. Since $0 = \{f_j, f_k\}_\omega = \omega(X_{f_j}, X_{f_k})$, the leaves of the associated (possibly singular) foliation are isotropic submanifolds and they are lagrangian at points where the functions are functionally independent. This is an example of a generalised real polarisation—i.e. an integrable distribution on TM whose leaves are lagrangian submanifolds, except for some singular leaves.

Definition 2.1. A real polarisation $\mathcal{P}$ is an integrable (in Sussmann's sense [5]) distribution of TM whose leaves are generically lagrangian. The complexification of $\mathcal{P}$ is denoted by P and will be called polarisation.

The most relevant polarisation for this article is $\langle X_{f_1}, \ldots, X_{f_n}\rangle_{C^\infty(M)}$: the distribution of the hamiltonian vector fields X_{f_i} of the components f_i of an integrable system $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$.

Definition 2.2. A point $p \in M$ in a $2n$ dimensional symplectic manifold (M, ω) is a nondegenerate singular point of the Williamson type (k_e, k_h, k_f) of an integrable system $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ if p is a critical point of rank $n - k_e - k_h - 2k_f$ and, in some symplectic system of coordinates $(x_1, y_1, \ldots, x_n, y_n)$ where $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$, the quadratic parts of $f_1, \ldots, f_n$ can be written as:

$$\begin{array}{lcr} h_i = x_i^2 + y_i^2 & \text{for } 1 \le i \le k_e, & \text{(elliptic)} \\ h_i = x_i y_i & \text{for } k_e + 1 \le i \le k_e + k_h, & \text{(hyperbolic)} \\ \begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1}, \\ h_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} & \begin{array}{c} \text{for } i = k_e + k_h + 2j - 1, \\ 1 \le j \le k_f \end{array} & \text{(focus–focus pair).} \end{array} \tag{1}$$

Example 2.1. For the simple pendulum the stable equilibrium point is an elliptic singularity, whilst the unstable one is of hyperbolic type. The spherical pendulum has a stable equilibrium point that is a purely elliptic singularity. The unstable equilibrium point is a focus–focus singularity. ◇

Here is an example of a real polarisation that does not come from an integrable system.

Example 2.2. The action of S^1 on $S^1 \times S^1$ given by $(z, x, y) \mapsto (z \cdot x, y)$, with $z, x, y \in S^1$, is symplectic (taking as symplectic form the area form of the torus). Because there are no fixed points, this action cannot be hamiltonian—otherwise, there would exist a function over a compact manifold without critical points. ◇

The first attempt to define quantum states was to see them as flat sections of a complex line bundle over the symplectic manifold, the so-called prequantum line bundle. The existence of global nonzero flat sections is a nontrivial matter, even when M is not compact. Actually, Rawnsley [4] (also Proposition 4.3 in this article, under slightly different hypotheses) showed that the existence of an S^1-action may be an obstruction for the existence of nonzero global flat sections.

Definition 2.3. A symplectic manifold (M, ω) such that the de Rham class $[\omega]$ is integral (it lies in the image of the homomorphism $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})$ induced by an inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$) is called prequantisable.

Definition 2.4. A prequantum line bundle of (M, ω) is a hermitian line bundle over M with connexion, compatible with the hermitian structure, (L, ∇^ω) that satisfies $curv(\nabla^\omega) = -i\omega$.

Example 2.3. Any exact symplectic manifold satisfies $[\omega] = 0$, in particular cotangent bundles with the canonical symplectic structure. The trivial line bundle is an example of a prequantum line bundle in this case. ◇

The following theorem (a proof can be found in [6]) provides a relation between the above definitions:

Theorem 2.1. *A symplectic manifold (M, ω) admits a prequantum line bundle (L, ∇^ω) if and only if it is prequantisable.*