Journal of Geometry and Physics 87 (2015) 461–481

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Explicit metrics for a class of two-dimensional cubically superintegrable systems

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ARTICLE INFO

Article history:
Received 4 March 2014
Received in revised form 31 July 2014
Accepted 4 August 2014
Available online 10 August 2014

Keywords:
Superintegrable Hamiltonian systems
Polynomially integrable Riemannian metrics on surfaces
Globally integrable Riemannian metrics on surfaces
Completely integrable systems on the 2-sphere

ABSTRACT

We obtain, in local coordinates, the explicit form of the two-dimensional, superintegrable systems of Matveev and Shevchishin involving linear and cubic integrals. This enables us to determine for which values of the parameters these systems are indeed globally defined on $\mathbb{S}^2$.

1. Introduction

The study of superintegrable dynamical systems has received many important developments reviewed recently in [1]. While integrable systems on the cotangent bundle T^*M of a n-dimensional manifold, M, require a set of functionally independent observables $(H, Q_1, \ldots, Q_{n-1})$ which are all in involution for the Poisson bracket $\{\cdot, \cdot\}$, a superintegrable system is made out of $\nu \geq n$ functionally independent observables

$$H, \qquad Q_1, \qquad Q_2, \qquad \cdots \quad Q_{\nu-1},$$

with the constraints

$$\{H, Q_i\} = 0, \quad \text{for all } i = 1, 2, \ldots, \nu - 1. \tag{1}$$

The maximal value of ν is $2n - 1$ since the system (1) reads $dH(X_{Q_i}) = 0$, implying that the span of the Hamiltonian vector fields, X_{Q_i}, is, at each point of T^*M, a subspace of the annihilator of the 1-form dH, the latter being of dimension $2n - 1$. Let us observe that for two-dimensional manifolds, a superintegrable system is necessarily maximal since $\nu = 3$.

As is apparent from [1], the large amount of results for superintegrable models is restricted to *quadratically* superintegrable ones, which means that the integrals Q_i are either linear or quadratic in the momenta, and the metrics on which these systems are defined are either flat or of constant curvature. For manifolds of non constant curvature, Koenigs [2] gave examples of quadratically superintegrable models. For some special values of the parameters the metrics happen to be defined on a manifold, M, which is never closed (compact without boundary).

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http://dx.doi.org/10.1016/j.geomphys.2014.08.004
0393-0440/

In their quest for superintegrable systems defined on closed manifolds, Matveev and Shevchishin [3] have given a complete classification of all (local) Riemannian metrics on surfaces of revolution, namely

$$G = \frac{dx^2 + dy^2}{h_x^2}, \qquad h = h(x), \qquad h_x = \frac{dh}{dx}, \tag{2}$$

which have a superintegrable geodesic flow (whose Hamiltonian will henceforth be denoted by H), with integrals $L = P_y$ and S respectively linear and cubic in momenta, opening the way to the new field of *cubically* superintegrable models. Let us first recall their main results.

They proved that if the metric G is not of constant curvature, then $\mathcal{I}^3(G)$, the linear span of the cubic integrals, has dimension 4 with a natural basis L^3, LH, S_1, S_2, and with the following structure. The map $\mathcal{L} : S \to \{L, S\}$ defines a linear endomorphism of $\mathcal{I}^3(g)$ and one of the following possibilities hold:

(i) $\mathcal{L}$ has purely real eigenvalues $\pm\mu$ for some real $\mu > 0$, then S_1, S_2 are the corresponding eigenvectors.
(ii) $\mathcal{L}$ has purely imaginary eigenvalues $\pm i\mu$ for some real $\mu > 0$, then $S_1 \pm iS_2$ are the corresponding eigenvectors.
(iii) $\mathcal{L}$ has the eigenvalue $\mu = 0$ with one Jordan block of size 3, in this case

$$\{L, S_1\} = \frac{A_3}{2} L^3 + A_1 LH, \qquad \{L, S_2\} = S_1,$$

for some real constants A_1 and A_3. Superintegrability is then achieved provided the function h be a solution of following non-linear first-order differential equations, namely

$$\begin{aligned}
&\text{(i) } h_x(A_0 h_x^2 + \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x) \\
&\text{(ii) } h_x(A_0 h_x^2 - \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sinh(\mu x)}{\mu} + A_4 \cosh(\mu x) \\
&\text{(iii) } h_x(A_0 h_x^2 - A_1 h + A_2) = A_3 x + A_4
\end{aligned} \tag{3}$$

and the explicit form of the cubic integrals was given in all three cases. For instance, when $\mu = 1$ or $\mu = i$, their structure is

$$S_{1,2} = e^{\pm\mu y}\left(a_0(x) P_x^3 + a_1(x) P_x^2 P_y + a_2(x) P_x P_y^2 + a_3(x) P_y^3\right), \tag{4}$$

where the $a_i(x)$ are explicitly expressed in terms of h and its derivatives; see [3].

For $A_0 = 0$ these equations are easily integrated and one obtains the Koenigs metrics [2], while the cubic integrals have the reducible structure $S_{1,2} = P_y Q_{1,2}$ where the quadratic integrals $Q_{1,2}$ are precisely those obtained by Koenigs.

Furthermore it was proved that in the case (ii), under the conditions

$$\mu > 0, \qquad A_0 > 0, \qquad \mu A_4 > |A_3|, \tag{5}$$

the metric and the cubic integrals are real-analytic and globally defined on $\mathbb{S}^2$.

The aim of this article is on the one hand to integrate explicitly the three differential equations in (3) and, on the other hand, to determine, by a systematic case study, all special cases which lead to superintegrable models *globally* defined on simply-connected, *closed*, Riemann surfaces.

In Section 2 we analyze the trigonometric case (real eigenvalues), integrating explicitly the differential equation (3)(i) to get an explicit local form for the metric and the cubic integrals. The global questions are then discussed, and we show that there is no closed manifold, M, on which the superintegrable model under consideration can be defined.

In Section 3 we investigate the hyperbolic case (purely imaginary eigenvalues). Here too, the integration of the differential equation (3)(ii) provides an explicit form for both the metric and the cubic integrals.

The previous results allows the determination of all superintegrable systems globally defined on $\mathbb{S}^2$, and these are proved in Theorems 1 and 2, namely

Theorem 1. *The metric*

$$G = \rho^2 \frac{dv^2}{D} + \frac{4D}{P} d\phi^2, \quad v \in (a, 1), \ \phi \in \mathbb{S}^1,$$

with

$$D = (v - a)(1 - v^2), \qquad P = (v^2 - 2av + 1)^2, \qquad -\rho = 1 + 4\frac{(v - a)D}{P}, \tag{6}$$

is globally defined on $\mathbb{S}^2$*, as well as the Hamiltonian*

$$H = \frac{1}{2} G^{ij} P_i P_j = \frac{1}{2}\left(\Pi^2 + \frac{P}{4D} P_\phi^2\right), \quad \Pi = \frac{\sqrt{D}}{\rho} P_v,$$

iff $a \in (-1, +1)$*. The two cubic integrals* S_1 *and* S_2*, also globally defined on* $\mathbb{S}^2$*, read*

$$S_1 = \cos\phi\, \mathcal{A} + \sin\phi\, \mathcal{B}, \qquad S_2 = -\sin\phi\, \mathcal{A} + \cos\phi\, \mathcal{B}, \tag{7}$$