Journal of Geometry and Physics 81 (2014) 128–144

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Matching groups and gliding systems

Vladimir Turaev *

Department of Mathematics, Indiana University, Bloomington IN 47405, USA

A R T I C L E   I N F O

Article history:
Received 19 February 2014
Accepted 20 March 2014
Available online 26 March 2014

MSC:
05C10
20F36
20F67
57M15
82B20

Keywords:
Graph
Cubed complex
Dimer covering
Matching
Perfect matching

A B S T R A C T

With every matching in a graph we associate a group called the matching group. We study this group using the theory of nonpositively curved cubed complexes. Our approach is formulated in terms of the so-called gliding systems.

1. Introduction

Consider a graph Γ without loops but possibly with multiple edges. A matching A in Γ is a set of edges of Γ such that different edges in A have no common vertices. Matchings are extensively studied in graph theory usually with the view to define numerical invariants of graphs. In this paper we study transformations of matchings determined by even cycles. An even cycle in Γ is an embedded circle in Γ formed by an even number of edges. If a matching A meets an even cycle s at every second edge of s, then removing these edges from A and adding instead all the other edges of s we obtain a new matching denoted sA. We say that sA is obtained from A by *gliding* along s. The inverse transformation is the gliding of sA along s which, obviously, gives back A. Composing the glidings, we can pass back and forth between matchings. If two even cycles s, t have no common vertices and a matching A meets both s and t at every second edge, then the compositions $A \mapsto sA \mapsto tsA$ and $A \mapsto tA \mapsto stA = tsA$ are considered as the same transformation. For any matching A in Γ, the compositions of glidings carrying A to itself form a group $\pi_A = \pi_A(\Gamma)$ called the *matching group*. Similar groups were first considered in [1] in the context of domino tilings of planar regions.

In the rest of the introduction, we focus on the matching groups in finite graphs. We prove that they are torsion-free, residually nilpotent, residually finite, biorderable, biautomatic, have solvable word and conjugacy problems, satisfy the Tits alternative, embed in $SL_n(\mathbb{Z})$ for some n, and embed in finitely generated right-handed Artin groups. Our main tool in the proof of these properties is an interpretation of the matching groups as the fundamental groups of nonpositively curved cubed complexes. The universal coverings of such complexes are Cartan–Alexandrov–Toponogov (0)-spaces in the sense of Gromov (CAT(0)-spaces). All necessary definitions from the theory of cubed complexes are recalled in the paper.

* Tel.: +1 8128556754.
E-mail address: vturaev@yahoo.com.

http://dx.doi.org/10.1016/j.geomphys.2014.03.009
0393-0440/

Using much more elementary considerations, we give a presentation of the matching group by generators and relations as follows. The set of vertices of a finite graph Γ adjacent to the edges of a matching A in Γ is denoted ∂A. We say that two matchings A, B in Γ are *congruent* if $\partial A = \partial B$. We explain that any tuple of matchings in Γ congruent to a given matching A_0 determines an element in π_{A_0}. The group π_{A_0} is generated by the elements $\{x_{A,B}\}_{A,B}$ associated with the 2-tuples A, B of matchings congruent to A_0. The defining relations: $x_{A_0,A} = 1$ for any A congruent to A_0 and $x_{A,C} = x_{A,B}\, x_{B,C}$ for any matchings A, B, C congruent to A_0 such that every vertex in ∂A_0 is incident to an edge which belongs to at least two of the matchings A, B, C. As a consequence, the group π_{A_0} is finitely generated and its rank is smaller than or equal to $M(M-1)/2$ where M is the number of matchings in Γ congruent to A_0 and distinct from A_0.

We define two families of natural homomorphisms between matching groups. First, any subset A' of a matching A in Γ is itself a matching in Γ. We define a canonical injection $\pi_{A'} \hookrightarrow \pi_A$. Identifying $\pi_{A'}$ with its image, one can treat $\pi_{A'}$ as a subgroup of π_A. Second, any two congruent matchings A, B in Γ may be related by glidings, and, as a consequence, their matching groups are isomorphic. We exhibit a canonical isomorphism $\pi_A \approx \pi_B$. We also relate the matching groups to the braid groups of graphs. This allows us to derive braids in graphs from tuples of matchings. We will briefly discuss a generalization of the matching groups to hypergraphs.

A special role in the theory of matchings is played by perfect matchings also called dimer coverings. A matching in a graph is perfect if every vertex of the graph is incident to a (unique) edge of this matching. Perfect matchings have been extensively studied in connection with exactly solvable models of statistical mechanics and with path algebras, see [2,3] and references therein. The matching groups associated with perfect matchings are called *dimer groups*. Since all perfect matchings in a finite graph are congruent, their dimer groups are isomorphic. The resulting isomorphism class of groups is an invariant of the graph.

The study of glidings suggests a more general framework of gliding systems in groups. A *gliding system* in a group G consists of certain elements of G called *glides* and a relation on the set of glides called *independence* satisfying a few axioms. Given a gliding system in G and a set $\mathcal{D} \subset G$, we construct a cubed complex $X_{\mathcal{D}}$ called the *glide complex*. The fundamental groups of the components of $X_{\mathcal{D}}$ are the *glide groups*. We formulate conditions ensuring that $X_{\mathcal{D}}$ is nonpositively curved. One can view gliding systems as devices producing nonpositively curved complexes and interesting groups. The matching groups and, in particular, the dimer groups are instances of glide groups for appropriate G and $\mathcal{D}$.

The paper is organized as follows. In Section 2 we recall the basics on cubed complexes and cubic maps. The next three sections deal with glidings: we define the gliding systems (Section 3), construct the glide complexes (Section 4), and study natural maps between the glide groups (Section 5). Next, we introduce dimer groups (Section 6), compute them via generators and relations (Section 7), and define and study the matching groups (Section 8). In Section 9 we consider connections with braid groups. In Section 10 we interpret the dimer complex in terms of graph labelings. In Section 11 we discuss the matching groups of hypergraphs. In the appendix we examine the typing homomorphisms of the matching groups.

2. Preliminaries on cubed complexes and cubical maps

We discuss the basics of the theory of cubed complexes and cubical maps, see [4], Chapters I.7 and II.5 for more details.

2.1. Cubed complexes

Set $I = [0, 1]$. A *cubed complex* is a CW-complex X such that each (closed) k-cell of X with $k \geq 0$ is a continuous map from the k-dimensional cube I^k to X whose restriction to the interior of I^k is injective and whose restriction to each $(k-1)$-face of I^k is an isometry of that face onto I^{k-1} composed with a $(k-1)$-cell $I^{k-1} \to X$ of X. The k-cells $I^k \to X$ are not required to be injective. The k-skeleton $X^{(k)}$ of X is the union of the images of all cells of dimension $\leq k$.

For example, the cube I^k together with all its faces is a cubed complex. So is the k-dimensional torus obtained by identifying opposite faces of I^k.

The *link* $LK(A) = LK(A; X)$ of a 0-cell A of a cubed complex X is the space of all directions at A. Each triple ($k \geq 1$, a vertex a of I^k, a k-cell $\alpha : I^k \to X$ of X carrying a to A) determines a $(k-1)$-dimensional simplex in $LK(A)$ in the obvious way. The faces of this simplex are determined by the restrictions of α to the faces of I^k containing a. The simplices corresponding to all triples (k, a, α) cover $LK(A)$ but may not form a simplicial complex. We say, following [5], that the cubed complex X is *simple* if the links of all $A \in X^{(0)}$ are simplicial complexes, i.e., all simplices in $LK(A)$ are embedded and the intersection of any two simplices in $LK(A)$ is a common face.

A *flag complex* is a simplicial complex such that any finite collection of pairwise adjacent vertices spans a simplex. A cubed complex is *nonpositively curved* if it is simple and the link of each 0-cell is a flag complex. A theorem of M. Gromov asserts that the universal covering of a connected finite-dimensional nonpositively curved cubed complex is a CAT(0)-space. Since CAT(0)-spaces are contractible, all higher homotopy groups of such a complex X vanish while the fundamental group $\pi = \pi_1(X)$ is torsion-free. This group satisfies a strong form of the Tits alternative: each subgroup of π contains a rank 2 free subgroup or virtually is a finitely generated abelian group, see [6]. Also, π does not have Kazhdan's property (T), see [7]. If X is compact, then π has solvable word and conjugacy problems and is biautomatic, see [8].