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On some dynamical and geometrical properties of the Maxwell–Bloch equations with a quadratic control

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ABSTRACT

In this paper, we analyze the stability of the real-valued Maxwell–Bloch equations with a control that depends on state variables quadratically. We also investigate the topological properties of the energy-Casimir map, as well as the existence of periodic orbits and explicitly construct the heteroclinic orbits.

1. Introduction

The description of the interaction between laser light and a material sample composed of two-level atoms begins with Maxwell's equations of the electric field and Schrödinger's equations for the probability amplitudes of the atomic levels. The resulting dynamics is given by Maxwell–Schrödinger equations which have Hamiltonian formulation and moreover there exists a homoclinic chaos [8].

Using the Melnikov method [14], in [9] the presence of special homoclinic orbits for the dynamics of an ensemble of two-level atoms in a single-mode resonant laser cavity with external pumping and a weak coherent probe modeled by Maxwell–Bloch's equations with a probe was established.

Fordy and Holm [7] discussed the phase space geometry of the solutions of the system introduced by Holm and Kovacic [8].

In 1992, David and Holm [6] presented the phase space geometry of the mentioned system restricted to $\mathbb{R}$, so named real-valued Maxwell–Bloch equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = xz \\ \dot{z} = -xy. \end{cases} \tag{1}$$

In 1996, Puta [18] considered system (1) with a linear, respectively a quadratic control u about Oy axis:

$$\begin{cases} \dot{x} = y \\ \dot{y} = xz + u \\ \dot{z} = -xy. \end{cases} \tag{2}$$

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These particular perturbations arise naturally in controllability context and were analyzed from the dynamical point of view. More precisely, in the case of the quadratic control $u = (k-1)xz$ with the parameter $k > 0$ [18], the dynamical analysis is done by proving that the restricted dynamics on each symplectic leaf of the associated Poisson configuration manifold is equivalent to the dynamics of the Duffing oscillator with control and with the pendulum dynamics.

In our work, we consider system (2), where $u = (k-1)xz$ with $k < 0$. We give a Poisson structure and we find a symplectic realization of the system. Using the method introduced in [20], we find the image of the energy-Casimir map and we study the topology of the fibers of the energy-Casimir map.

For details on the Poisson geometry and the Hamiltonian mechanical system, see, e.g. [5,13,17,12].

2. Poisson structure, symplectic realization and geometric prequantization

Considering the quadratic parametric control $u = (k-1)xz$, system (2) becomes:

$$\begin{cases} \dot{x} = y \\ \dot{y} = kxz \\ \dot{z} = -xy, \end{cases} \tag{3}$$

where $k < 0$ is the tuning parameter, according to the classification of chaos control methods [3,4].

The constants of motion

$$H_k(x, y, z) = \frac{1}{2}(y^2 + kz^2), \qquad C(x, y, z) = \frac{1}{2}x^2 + z$$

were given in [18]. Using the Euclidean space $\mathbb{R}^3$ with a modified cross-product as a Lie algebra, a Poisson structure Π,

$$\Pi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & x \\ -0 & -x & 0 \end{bmatrix}$$

was also given.

We are going to give a Lie algebra, isomorphic with that mentioned above, on its dual space the same Poisson structure is obtained.

Let us consider the Heisenberg Lie group H_3,

$$H_3 = \left\{ A \in GL(3, \mathbb{R}) \,|\, A = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix},\ a, b, c \in \mathbb{R} \right\}.$$

The corresponding Lie algebra h_3 is

$$h_3 = \left\{ X \in gl(3, \mathbb{R}) \,|\, X = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & c & 0 \end{bmatrix},\ a, b, c \in \mathbb{R} \right\}.$$

Note that, as a real vector space, h_3 is generated by the base $B_{h_3} = \{E_1, E_2, E_3\}$, where

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following bracket relations $[E_1, E_2] = 0$, $[E_1, E_3] = 0$, $[E_2, E_3] = E_1$, hold.

Following [12], it is easy to see that the bilinear map $\Theta : h_3 \times h_3 \to \mathbb{R}$ given by the matrix $(\Theta_{ij})_{1 \le i,j \le 3}$, $\Theta_{12} = -\Theta_{21} = 1$ and 0 otherwise, is a 2-cocycle on h_3 and it is not a coboundary since $\Theta(E_1, E_2) = 1 \neq 0 = f([E_1, E_2])$, for every linear map f, $f : h_3 \to \mathbb{R}$.

On the dual space $h_3^* \simeq \mathbb{R}^3$, a modified Lie–Poisson structure is given in coordinates by

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -x & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & x \\ 0 & -x & 0 \end{bmatrix}.$$

The function H_k is the Hamiltonian and C is a Casimir of our configuration.

The next proposition states that system (3) can be regarded as a Hamiltonian mechanical system.

Proposition 1. *The Hamilton–Poisson mechanical system $(\mathbb{R}^3, \Pi, H_k)$ has a full symplectic realization $(\mathbb{R}^4, \omega, \tilde{H}_k)$, where*

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

and

$$\tilde{H}_k = \frac{1}{2}\left(p_1^2 + kp_2^2 - kp_2q_1^2 + \frac{k}{4}q_1^4\right).$$