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Metric Tannakian duality ☆

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ABSTRACT

We incorporate metric data into the framework of Tannaka–Krein duality. Thus, for any group with left invariant metric, we produce a dual metric on its category of unitary representations. We characterize the conditions under which a "double-dual" metric on the group may be recovered from the metric on representations, and provide conditions under which a metric agrees with its double-dual. We also explore a diverse class of possible applications of the theory, including applications to T-duality and to quantum Gromov–Hausdorff distance.

1. Introduction

In this article we study a duality between length functions on groups and distances between their associated group representations. Thus to begin with, we describe how a metric on a group induces a notion of distance between its representations, and we show that conversely, a distance function on a group's representations induces a (semi-)metric on the group.

We then investigate the obvious "double-dual" metric one obtains on a group by applying the two machines in succession, and in the course of this we characterize the conditions under which the original metric agrees with its double-dual. *A priori*, we had no expectations that our method of encoding group metrics in the category of representations would preserve any information at all. But it turns out that it does preserve information, in fact there are large classes of metrics for which the double-dual agrees with the original metric, and furthermore, even when it does not, a stabilization occurs with double-dualizations: the double-dual and the double–double-dual metric are always equal.

In the case of compact topological groups, these constructions can be viewed as the incorporation of metric data into the Tannaka duality [1]:

$$G \longleftrightarrow \mathrm{Reps}(G)$$

between a group and its category of finite dimensional complex representations.

Alternatively, one might view the constructions as a group theoretic analog of the duality introduced by Connes [2]:

$$(X, d) \longleftrightarrow (C(X), L_d)$$

between a locally compact metric space (X, d) and the commutative C^*-algebra with Lip-norm $(C(X), L_d)$.

In our view the theory is interesting because of two types of applications. First, there exist subtle relationships between groups which can only be expressed as equivalences of categories associated to them (we have in mind T-duality, Fourier–Mukai equivalence, and some cases of mirror symmetry), so if metric data can be encoded in the relevant categories,

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it can be transported across the equivalences. The second (and more speculative) application is that Rieffel's quantum Gromov–Hausdorff distance [3] can be used to define a metric on the representations of a compact Lie group. Comparing this with our type of metric on representations, one might extract new information about quantum-Gromov–Hausdorff convergence. We will consider some of these applications in Section 5.

The outline of the note is as follows. In Section 2 we define the distance between two representations of a metric group. This distance depends on a certain choice, and we examine two such choices—one which is best suited for finite dimensional Lie group representations, and another which applies for arbitrary representations. In Section 3 we define the double-dual metric. We then generalize this in the spirit of Tannaka–Krein duality: for a category $\mathcal{T}$ with a metric on its objects and a functor to Hilbert spaces $\mathcal{T} \xrightarrow{F} \mathrm{Hilb}$, we produce a (possibly infinite) semi-metric on the group $\mathrm{Aut}\, F$ of natural equivalences of F. The last section is devoted to examples and applications.

2. The metric on representations

Let G be a group with *semi-length* function ℓ. Thus ℓ is a function $G \to [0, \infty]$ satisfying $\ell(id_G) = 0$, $\ell(g) = \ell(g^{-1})$, and $\ell(gg') \leq \ell(g) + \ell(g')$. Semi-length functions correspond to left invariant semi-metrics on G via

$$\ell(g) = d(id_G, g) \longleftrightarrow d(g, g') = \ell(g^{-1}g').$$

If a semi-length function vanishes only at the identity then it is called a *length* function; length functions correspond to left invariant metrics. By default our semi-lengths and semi-metrics are allowed to take infinite values, but if they do not we refer to them as *finite*.

Now let us define the distance $d^{\mathrm{Reps}}(\rho, \sigma)$ between two representations of a group with semi-length function. The result will depend on a choice, which we make now, of an infinite dimensional Hilbert space $\mathfrak{h}$ and a bi-invariant metric $d^{U(\mathfrak{h})}$ on its group $U(\mathfrak{h})$ of unitary operators. Two such choices are examined at the end of the section.

First, given two representations $\rho, \sigma : G \to U(\mathfrak{h})$, form the distance:

$$\overline{d}(\rho, \sigma) := \sup_{\ell(g)\neq 0} \frac{d^{U(\mathfrak{h})}(\rho_g, \sigma_g)}{\ell(g)}.$$

This function is symmetric and bi-invariant because $d^{U(\mathfrak{h})}$ is so, and it satisfies the triangle inequality as follows immediately from:

$$\sup_{\ell(g)\neq 0} \frac{d^{U(\mathfrak{h})}(\rho_g, \tau_g) + d^{U(\mathfrak{h})}(\tau_g, \sigma_g)}{\ell(g)} \leq \sup_{\ell(g)\neq 0} \frac{d^{U(\mathfrak{h})}(\rho_g, \tau_g)}{\ell(g)} + \sup_{\ell(g')\neq 0} \frac{d^{U(\mathfrak{h})}(\tau_{g'}, \sigma_{g'})}{\ell(g')}.$$

But this is not yet the desired semi-metric for representations. We would like to enforce that unitarily equivalent representations have zero distance from each other. Also, we need a notion of distance between two representations whose underlying Hilbert spaces are not equal. Thus, for representations $\rho : G \to U(V_\rho)$ and $\sigma : G \to U(V_\sigma)$, we set

$$d^{\mathrm{Reps}}(\rho, \sigma) := \inf_{V_\rho \overset{a}{\hookrightarrow} \mathfrak{h},\ V_\sigma \overset{b}{\hookrightarrow} \mathfrak{h}} \overline{d}(a_*\rho, b_*\sigma). \tag{2.1}$$

Here a, b label the (not necessarily surjective) isometries into $\mathfrak{h}$, and $a_*\rho$ denotes the representation on $\mathfrak{h}$ obtained by "adding 1's on the diagonal". (More precisely, decompose $\mathfrak{h}$ as $\mathfrak{h} = a(V_\rho) \oplus a(V_\rho)^\perp$, then $a_*(\rho)|_{a(V_\rho)} := a\rho a^{-1}$ and $a_*(\rho)|_{a(V_\rho)^\perp} := id_{a(V_\rho)^\perp}$.)

But once two isometries $V_\rho \overset{a}{\hookrightarrow} \mathfrak{h}$ and $V_\sigma \overset{b}{\hookrightarrow} \mathfrak{h}$ have been chosen, all other isometries can be obtained by composing with elements of $U(\mathfrak{h})$. Thus it is clear that

$$d^{\mathrm{Reps}}(\rho, \sigma) \equiv \inf_{u,v\in U(\mathfrak{h})} \overline{d}(u(a_*\rho)u^*, v(b_*\sigma)v^*). \tag{2.2}$$

Of course this does not depend on the initial choice of a and b, so they will usually be omitted. Also, because $\overline{d}$ is bi-invariant it is only necessary to take the infimum in one variable. Thus

$$d^{\mathrm{Reps}}(\rho, \sigma) \equiv \inf_{u\in U(\mathfrak{h})} \overline{d}(a_*\rho, u(b_*\sigma)u^*). \tag{2.3}$$

Note that if the Hilbert space V_ρ of some representation has dimension greater than $\dim(\mathfrak{h})$ so that there are no isometries $V_\rho \to \mathfrak{h}$, then Eq. (2.1) is an infimum over the empty set which equals $+\infty$.[1] So representations of dimension greater than $\dim(\mathfrak{h})$ have infinite distance from all other representations. They do not contribute any information for the purposes of metric duality, and we will tacitly exclude them.

With this in mind, we fix the following notation: $\mathrm{Reps}(G)$ denotes the category whose objects are unitary representations of G on Hilbert spaces and whose arrows are G-module maps (arrows are automatically isometric). $\mathrm{Reps}_{\mathrm{fin}}(G)$ denotes the subcategory of finite dimensional representations.

[1] Infimums and supremums in our context are computed on subsets of $[0, +\infty]$, thus the infimum of the empty set is $+\infty$, while the supremum of the empty set is 0.