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Frobenius manifold structures on the spaces of abelian integrals

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ABSTRACT

Frobenius manifold structures on the spaces of abelian integrals were constructed by I. Krichever. We use $\mathcal{D}$-modules, deformation theory, and homological algebra to give a coordinate-free description of these structures. It turns out that the tangent sheaf multiplication has a cohomological origin, while the Levi-Civita connection is related to one-dimensional isomonodromic deformations.

1. Introduction

Frobenius manifolds are manifolds with a flat metric and a multiplication in the tangent sheaf, subject to some constraints. Frobenius manifolds were introduced by Dubrovin in [1,2] as a mathematical framework for deformations of topological quantum field theories (see also [3]). In mathematics Frobenius manifolds arise in two different situations, corresponding to A-models and B-models in physics. In an A-model one counts rational curves on a variety; this is also known as Gromov–Witten invariants. The generating function for these invariants is the potential for the corresponding Frobenius manifold.

This paper is concerned with B-models. In a B-model one studies deformations of a certain complex structure (formal or analytic). The best known examples are extended moduli spaces of Calabi–Yau varieties [4] and the unfoldings of isolated singularities [5] (see [6] for an exposition). We would like to mention that Frobenius structures are important for mirror symmetry: if two varieties are mirror dual to each other, then the A-model Frobenius manifold, corresponding to the first variety, is isomorphic to the B-model Frobenius manifold, corresponding to the second.

1.1. Moduli spaces of abelian integrals

Examples of Frobenius manifolds are furnished by Hurwitz spaces. Hurwitz spaces parameterize pairs (X, f), where X is a smooth complete algebraic curve, $f : X \to \mathbb{P}^1$. Dubrovin constructed Frobenius structures on Hurwitz spaces [3].

Our main object is the following deformation of a Hurwitz space: a space of pairs (X, f), where f is a *multi-valued* function such that df is a single-valued meromorphic 1-form with prescribed periods and residues. If the periods and residues are equal to zero, then this space is a Hurwitz space. Our spaces will be called *spaces of abelian integrals*.

Krichever constructs in [7,10] Frobenius structures on the universal covers of the spaces of abelian integrals. Our main goal is to give a coordinate-free geometric description of these Frobenius structures. We also generalize the setup to the

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case of multiple poles and non-zero residues in Section 5. In particular, our generalization covers the previously untreated case of abelian integrals of the third kind. Our approach is based on a $\mathcal{D}$-module push-forward (also known as twisted de Rham complex; see [6, Section I.3.3]). It turns out that these structures of Frobenius manifolds have a nice interpretation: the tangent sheaf multiplication has a cohomological origin, similar to that of [4]. The metric and the Levi-Civita connection are closely related to one-dimensional isomonodromic deformation. (This is not directly related to isomonodromic deformations used to describe the semi-simple Frobenius manifolds.)

We are using the approach to Frobenius structures via primitive forms. This has been invented by Saito [5]. We would like to mention a striking similarity between three constructions of Frobenius structures: on the universal unfolding of isolated singularity [5], on the extended moduli space of Calabi–Yau varieties [4], and our construction. In each case a pencil of connections is obtained by the (derived) direct image. Our case is, in some sense, intermediate: on the one hand, singularities are present, on the other hand, our structure is not local, it depends on the global geometry of a curve. This is why we hope that, generalizing our construction to higher dimension, we shall provide a bridge between the pictures of Saito and Barannikov–Kontsevich, giving a unified approach to B-model Frobenius manifolds.

Another interesting feature of our construction is that we get a *family* of Frobenius manifolds parameterized by the periods of abelian integrals. We also want to emphasize that there are some new features specific for the higher genus case: to get a Frobenius structure we need to make a modification of the direct image (see Section 3.3).

2. Preliminaries and the main construction

2.1. Pencils of connections

Definition 1. Let $p_1 : \mathbf{M} \times \mathbb{P}^1 \to \mathbf{M}$ be the natural projection, where $\mathbf{M}$ is a manifold. By a *pencil of connections* on $\mathbf{M}$ we mean a pair $(\mathcal{W}, \nabla)$, where $\mathcal{W} \to \mathbf{M}$ is a vector bundle, ∇ is a relative flat connection on $\mathcal{V} = p_1^*\mathcal{W}$ along $\mathbf{M}$ with a simple pole along $\mathbf{M} \times \{0\}$.

One interprets a pencil as a family of flat connections on $\mathcal{W}$, parameterized by $\mathbb{P}^1 \setminus 0$. The condition on the pole implies that this family is of the form $\nabla_\infty + \Phi/z$, this is why it is called "pencil" (here z is a coordinate on $\mathbb{P}^1$). There is a natural way to construct twisted Frobenius manifold structures on dense open subsets of $\mathbf{M}$ starting from a pencil of connections, provided this pencil of connections satisfies some non-degeneracy condition (and, conversely, every Frobenius manifold gives rise to a pencil of connections). This will be explained in detail in Section 4.1.

2.2. Main objects

Consider a smooth complete algebraic curve X of genus g over $\mathbb{C}$, let $p \in X$. Denote by $(\hat{X}, \hat{p})$ the maximal abelian cover of (X, p).

Definition 2. An *abelian integral* on (X, p) is a function f on $\hat{X}$ such that df descends to a meromorphic differential on X. We define *periods* of f to be those of df.

Remarks. (1) An abelian integral can be thought as an integral of a meromorphic form on X. Thus the space of abelian integrals is a one-dimensional affine bundle over the vector bundle of differential forms.

(2) One can avoid working with abelian integrals by fixing a point $p_0 \in \hat{X}$. Then the affine bundle trivializes, a section being the space of abelian integrals that vanish at p_0, see also Section 5.

To simplify notation, we shall assume first that df has a single pole. We outline the changes needed in the multi-pole case in Section 5.

Let $n \geq 1$ be an integer. Consider the moduli space $\mathbf{A}_{g,n}$ of triples (X, p, f), where (X, p) is as above, f is an abelian integral with a single pole of order exactly n at p (in other words, df is a meromorphic form on X with the only pole at p of order $n + 1$).

The periods of f give a linear map $H_1(X, \mathbb{Z}) \to \mathbb{C}$. One can identify groups $H_1(X, \mathbb{Z})$ locally over the moduli space of curves using the Gauss–Manin connection, therefore the periods give rise to a foliation on $\mathbf{A}_{g,n}$. Let us fix one of the leaves and denote its smooth locus by $\mathbf{A}$. Thus, roughly speaking, $\mathbf{A}$ parameterizes abelian integrals with prescribed periods.

Let $\hat{\mathbf{A}}$ be the moduli space of quadruples (X, p, f, Δ), where $(X, p, f) \in \mathbf{A}$, Δ is a subgroup of $H_1(X, \mathbb{Z})$ maximal isotropic with respect to the intersection form. The elements of Δ will be called *a-cycles*. Clearly, $\hat{\mathbf{A}}$ is a cover of $\mathbf{A}$. Our main result is the following

Theorem. (a) *There is a natural pencil of connections on* $\hat{\mathbf{A}}$.

(b) *This pencil of connections gives rise to a twisted Frobenius structure on* $\hat{\mathbf{A}}$.