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Relationship of *d*-dimensional continuous multi-scale wavelet shrinkage with integro-differential equations

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Abstract

The goal of this paper is to extend the results of Didas and Weickert [Didas, S, Weickert, J. Integrodifferential equations for continuous multi-scale wavelet shrinkage. Inverse Prob Imag 2007;1:47–62.] to d-dimensional ($d \geqslant 1$) case. Firstly, we relate a d-dimensional continuous mother wavelet $\psi(x)$ with a fast decay and n vanishing moments to the sum of the order partial derivative of a group of functions $\theta^k(x)(|k| = n)$ with fast decay, which also makes wavelet transform equal to a sum of smoothed partial derivative operators. Moreover, d-dimensional continuous wavelet transform can be explained as a weighted average of pseudo-differential equations, too. For $d = 1$, our results are completely same as Didas and Weickert (2007), but for $d > 1$, it is different from the type of one variable. Finally, we exploit the reason with an example of 2-dimensional and 3-dimensional Mexican hat wavelet.

1. Introduction

From the viewpoint of approximation theory and harmonic analysis, the wavelet theory was important on several counts. It gave simple and elegant unconditional bases (wavelet bases) for function spaces (Lebesgue, Hardy, Sobolev, Besov, Triebel–Lizorkin) that simplified some aspects of Littlewood–Paley theory. It provided a very suitable vehicle for the analysis of the core linear operators of harmonic analysis and partial differential equations (Calderon–Zygmund theory). Moreover, it allowed the solution of various functional analytic and statistical extreme problems to be made directly from wavelet coefficients.

However, the great impetus came from the prominent applications in engineering fields, such as in image processing, in statistical estimation and some applications in physics both in the numerical and analytic treatment of differential equations.

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Given an image, the generic problems of image processing are compression, noise reduction, feature extraction and object recognition, of which wavelet shrinkage plays an important role.

In physics, many problems can be eventually presented in the form of partial differential equation (PDE). For example, as is well known, Maxwell equation is the basis of the theory of electromagnetic fields, which is a mathematical description of the classic electromagnetic theory created by Maxwell by summing up the fundamental laws of electromagnetism available as well as pulsing his own concept of displacement current. Generally, to begin with, an analysis of all electromagnetic processes can be made from Maxwell equations. So, the representation of Maxwell equations and the fast solving appear more and more important.

It is notable that the nonlinear wavelet methods for electromagnetic problems have made great progress, and have become more and more effective for Maxell equations. For example, G. Kaiser first introduced electromagnetic wavelets (also called physical wavelets) in a series of papers [20–22] as localized solutions of Maxwell's equations, which were used to analyse the construction of the solution spaces and properties, and had good effects on some applications to the scattering problems of both the electromagnetic waves and the acoustic waves. Although called wavelets, electromagnetic wavelets, in principle, are different from the compactly supported orthogonal wavelets defined by Daubechies, which are actually a group of operators, unlike the wavelet operators spoken in general.

One of us presented the new concepts of the frame operator group and the wavelet operator group in [23]. On the one hand, they are the generalization of conventional frame and wavelet, as well as the further analysis of the frame and wavelet operators. On the other hand, this theory easily unites the electromagnetic wavelets and the general wavelet functions. Moreover, he proved that the continuous wavelet transform belongs to the wavelet operator group, and found that the electromagnetic wavelet presented by Kaiser is the conjugate of the wavelet operator group. For more details and its application, see [23, chapter 2, p. 11–23].

On the other hand, a very interesting thing about wavelet shrinkage (see [1–8]) is that it can be motivated from very different fields of mathematics, namely partial differential equations, the calculus of variations, harmonic analysis or statistics, which, conversely, gives a great impetus to the applications of wavelet to many engineering fields. Of these, Donoho et al. analyzed wavelet shrinkage methods in the context of minimax estimation and showed that wavelet shrinkage generates asymptotically optimal estimates or noisy data that outperform any linear estimator [1–3]. At the same time, Devore and Lucier studied wavelet shrinkage in terms of minimization problems with the help of K-functional in [4,5].

Moreover, Lorenz [6] addresses and presents the multiple connections between different fields of mathematics: harmonic analysis, functional analysis, partial differential equations, or statistics with the wavelet shrinkage methods, that is, (1) both soft shrinkage of the discrete wavelet coefficients, the discrete Fourier coefficients and continuous Fourier transform, except the continuous wavelet transform, are the negative subgradient of a certain functional; (2) the discrete wavelet shrinkage is formulated as the minimization of a certain functional in Besov spaces; (3) the discrete wavelet shrinkage can also be seen as the projection onto convex sets in Hilbert spaces; and (4) the discrete wavelet shrinkage is equivalent to Bayesian de-noising. For more details see [6] and references therein.

Bredies et al. [7] show the equivalence of variation wavelet shrinkage to abstract pseudo-differential evolution equation. For discrete wavelet the equivalent results between the methods of both classes have also been shown under certain conditions [8], while it takes only the finest one into consideration other than the coarser scales.

As it is well known, the continuous wavelet transform has many advantages over the discrete wavelet transform, such as the weaker limited condition for the generate function, the arbitrary choice of the scale and translation, especially the wide application in the fields of pattern recognition, feature extraction and detection.

Further improvements of the understanding of wavelet shrinkage are due to the works [9,10], where the continuous wavelet shrinkage and translation invariant wavelet shrinkage are interpreted as smoothing scale spaces. For further discussion, we refer the reader to [11] and the papers referenced therein, for a more complete description of the properties of wavelet shrinkage.

Based on these ideas, [11] focuses on 1D signals and analyses the continuous shrinkage framework. Using the idea of understanding wavelets as smoothed derivative operators, they describe 1D continuous wavelet shrinkage as approximation to a novel integro-differential evolution equation and compare the corresponding energy functional that uses both smoothed derivative operators within the penalties and integration over all scales, with the classical regularization ones.

The key point of [11] is the fact that for 1-dimensional wavelets with a finite number of vanishing moments, the wavelet transform can be understood as applying a smoothed derivative operator [12, p. 167]. However, it is not the case for d-dimensional wavelets ($d \geqslant 2$) anymore. One natural question arises whether this result is right for higher dimension or not, since image is at least 2-dimensional. In fact, the connection of these pseudo-differential equations to well-known image processing methods such as nonlinear diffusion of Perona–Malik type [13] is not included in [11].