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Impulsive synchronization of Lipschitz chaotic systems

Yen-Sheng Chen [a], Chien-Cheng Chang [a,b,*]

[a] Division of Mechanics, Research Center for Applied Sciences, Academia Sinica, Taipei 115, Taiwan, ROC
[b] Institute of Applied Mechanics & Taida Institute of Mathematical Sciences, National Taiwan University, Taipei 106, Taiwan, ROC

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Abstract

Impulsive method is suitable for digital implementation of secure communication based on chaos synchronization. In the present study, it is assumed that the system satisfies the local Lipschitz condition where a Lipschitz constant is estimated a priori. An impulsive controller is shown to achieve synchronization of chaotic systems in the sense of exponential stability under one restriction relation (criterion). The Duffing two-well and the Rössler systems were simulated to illustrate the theoretical analysis.

1. Introduction

Chaotic systems were thought difficult to be synchronized or controlled in the past as chaotic systems exhibit sensitive dependence on initial conditions. Since 1980s, researchers have realized that chaotic motions can be synchronized through a feedback mechanism [1] or by linking two systems with common signals [2], as well as controlling chaos is also possible [3,4]. In the past decade, we have seen a rapid growth of theoretical and experimental studies on chaos synchronization. This is partly because chaos synchronization has potential applications in secure communication [5], information processing [6], pattern formation [7], etc.

Synchronization means that the state of the response system eventually approaches that of the driving system. Two kinds of chaos synchronization are most often discussed: (1) The master–slave scheme, introduced by Pecora and Carroll [2], consists of two identical systems with different initial conditions. The master system is chaotic while some state variables of the slave system are replaced by the corresponding ones of the master system. Synchronization occurs if and only if all the (conditional) Lyapunov exponents, quantifying stretching properties of some trajectories [8], to the unreplaced state variables are negative [2,9]. (2) The second kind of synchronization, the coupling scheme, deals with two identical chaotic systems except coupling terms which can be either unidirectional or bidirectional. Under certain conditions the response system may eventually evolve into the same orbit of the driving system.

Synchronization discussed above is called complete synchronization or simply synchronization. There are other types of synchronization such as generalized synchronization, phase synchronization, lag, and anticipated synchronization.

* Corresponding author.
E-mail addresses: ysc@gate.sinica.edu.tw (Y.-S. Chen), mechang@gate.sinica.edu.tw (C.-C. Chang).

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Generalized synchronization means that evolving orbits of the response system approach the image of those to the driving system under a function relationship [10,11]. This function is not necessarily defined on the whole phase space but on its attractor only. Phase synchronization is that the phases of two systems come closely [12,13] as time evolves though the amplitudes remain almost uncorrelated. Lag (anticipated) synchronization means that the state of the response system approaches eventually that of the driving system with a time delay [13] (lead [14]). Furthermore, synchronization of time-delayed systems and related issues can also be found in the literatures [15,16].

In this work, we propose an impulsive controller to achieve synchronization of chaotic systems. Impulsive control method fits digital implementation of secure communication based on chaos synchronization. It is therefore important to understand how impulsive control accomplishes chaos synchronization. This method is not only applied to chaos synchronization but also to medicine, biology, and economics. Some other impulsive designs, such as comparison method [17,18] and linear feedback [19] may also achieve synchronization of chaotic systems.

The systems considered in the present study are assumed to satisfy the local Lipschitz condition where a Lipschitz constant is estimated a priori. Next, impulse intervals and a feedback gain matrix are elected according to one single restriction relation (criterion) which connects the Lipschitz constant, the rate of convergence, the gain matrix, and impulse intervals. The estimated error state vector is proven to converge in the sense of exponential stability. The criterion is easy to implement in locating the admissible region of control parameters to ensure the occurrence of synchronization. Finally, the Duffing two-well and the Rössler systems are simulated to illustrate the validity of the theoretical analysis.

2. Theoretical analysis

Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{1}$$

and an impulsive controlled system

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(t, \hat{\mathbf{x}}), t \neq t_i^+, \tag{2a}$$

$$\Delta\hat{\mathbf{x}}|_{t=t_i} = \hat{\mathbf{x}}(t_i^+) - \hat{\mathbf{x}}(t_i^-) =: U(i), t = t_i^+, \tag{2b}$$

where $\mathbf{x}, \hat{\mathbf{x}} \in \mathbb{R}^N$ denote the state vectors, $\hat{\mathbf{x}}(t_0^+) = \hat{\mathbf{x}}_0$ and Ω is a domain containing the origin. The function $\mathbf{f} : \Omega \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the Lipschitz condition $\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leqslant L\|\mathbf{x}_1 - \mathbf{x}_2\|$ for all $(t, \mathbf{x}_1)$ and $(t, \mathbf{x}_2)$ in Ω with a Lipschitz constant L. The time sequence $\{t_i\}$ of impulse instants is $0 < t_1 < t_2 < \cdots < t_i < \cdots$, and $t_i \to \infty$ as $i \to \infty$. The present work requires

$$|t_i - t_{i-1}| \leqslant T < \infty, \quad i = 1, 2, \ldots. \tag{3}$$

Let the state error be $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$, then

$$\Delta\mathbf{e}(t_i) := [\hat{\mathbf{x}}(t_i^+) - \mathbf{x}(t_i^+)] - [\hat{\mathbf{x}}(t_i^-) - \mathbf{x}(t_i^-)] = \Delta\hat{\mathbf{x}}|_{t=t_i},$$

for $i = 1, 2, \ldots$. Notice that system (1) is continuous, $\mathbf{x}(t_i^+) - \mathbf{x}(t_i^-)$ is a zero vector at $t = t_i$ for $i = 1, 2, \ldots$. The error dynamical system becomes

$$\dot{\mathbf{e}} = \mathbf{f}(t, \mathbf{x} + \mathbf{e}) - \mathbf{f}(t, \mathbf{x}), t \neq t_i^+, \tag{4a}$$

$$\Delta\mathbf{e}(t_i) = \Delta\hat{\mathbf{x}}|_{t=t_i}, t = t_i^+, \quad i = 1, 2, \ldots. \tag{4b}$$

As a result, $\Delta\mathbf{e}(t_i) = U(i)$ at $t = t_i^+$ for $i = 1, 2, \ldots$. The purpose is to choose an appropriate impulsive controller $U(i)$ such that the error state $\mathbf{e}(t) = 0$ is asymptotically stable, i.e. system (2) synchronizes to system (1).

In the present study, we choose the impulsive controller in the form of negative feedback as

$$U(i) = \Gamma(i)\mathbf{e}(t_i^-), \quad i = 1, 2, \ldots, \tag{5}$$

where the gain matrix $\Gamma(i)$ actuated at $t = t_i^+$ is to be determined. A criterion of chaos synchronization for systems (1) and (2) is provided by the following theorem.

Theorem. *The error state* $\mathbf{e} = 0$ *is exponentially stable if the impulsive controller is* $U(i) = \Gamma(i)\mathbf{e}(t_i^-)$, *where* $\Gamma(i)$ *satisfies*

$$\|I + \Gamma(i)\| \leqslant \mathrm{e}^{-\alpha/2 - L(t_i - t_{i-1})}, \tag{6}$$

with $\alpha > 0$.