Available online at www.sciencedirect.com

ScienceDirect

Chaos, Solitons and Fractals 40 (2009) 1520–1529

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Variational iteration method for solving partial differential equations with variable coefficients

A.H.A. Ali [a,*], K.R. Raslan [b]

[a] Mathematics Department, Faculty of Science, Menoufia University, Shebein El-Koom, Egypt
[b] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt

Accepted 8 August 2007

Communicated by Prof. Ji-Huan He

Abstract

An extremely simple and elementary but rigorous derivation of exact solutions of partial differential equations in different dimensions with variable coefficients is given using the variational iteration method. The efficiency of the considered method is illustrated by some examples. The results show that the proposed iteration technique, without linearization or small perturbation, is very effective and convenient.

1. Introduction

A physically meaningful mathematical derivation of exact solutions of partial differential equations in different dimensions with variable coefficients seemed to be something which should come at the end of physical understanding of almost everything. This plausible attitude may have contributed to the fact that until very recently no a universal approach existed, thus the study of numerical methods for the solutions of various partial differential equations (PDEs) has enjoyed an intense period of activity over the last 40 years from both theoretical and practical points of view. Improvements in numerical techniques, together with the rapid advance in computer technology, have meant that many of the PDEs arising in engineering and scientific applications, which were previously intractable, can now be routinely solved [1]. The Adomian decomposition method, which was developed by Adomian [2], depends only on the initial conditions and results in a solution in series which converges to the exact solution of the problem. In recent years, other ansatz methods have been developed, such as the exp-function method [3], variational approaches [4,5], the tanh method [6–8], extended tanh function method [9,10], the modified extended tanh function method [11,12], the first integral method [13–17], and others [18]. Ansatz methods cannot applied to solve the partial differential equations in different dimensions with variable coefficients in x, y, and z, also the variational iteration method based on the initial conditions, but the ansatz does not depend on the initial conditions.

* Corresponding author. Tel.: +20 (02)22871081; fax: +20 (02)22744159.
E-mail addresses: ahaali_49@yahoo.com (A.H.A. Ali), kamal_raslan@yahoo.com (K.R. Raslan).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2007.09.031

The variational iteration method (VIM) [18–22] gives rapid convergent successive approximations of the exact solution if such a solution exists without any restrictive assumption or transformation that may change the physical behavior of the problem. The VIM gives several successive approximations through an iteration formulation of a correction functional. Moreover, the VIM reduces the size of calculation, while the tedious calculation is required for Adomian polynomials; hence the iteration is direct and straightforward.

In this paper, we use the VIM to find exact solutions of the PDEs in one, two and three dimensions with variable coefficients which will be useful in numerical studies. We hope that we can simplify the derivation procedure and reduce the mathematics to a bare minimum which as the reader will see, does not exceed a few line of elementary arithmetic.

2. The variational iteration method [18–22]

To illustrate the basic idea of the VIM we consider the following general PDE:

$$L_t u + L_x u + L_y u + L_z u + Nu = g(x,y,z,t). \tag{1}$$

where L_t, L_x, L_y and L_z are linear operators of t, x, y, and z, respectively, and N is a nonlinear operator. According to the VIM, we can express the following correction functional in t-, x-, y- and z-directions, respectively, as follows:

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda_1\{L_s u_n + (L_x + L_y + L_z + N)\tilde{u}_n - g\}\mathrm{d}s, \tag{2a}$$

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^x \lambda_2\{L_s u_n + (L_t + L_y + L_z + N)\tilde{u}_n - g\}\mathrm{d}s, \tag{2b}$$

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^y \lambda_3\{L_s u_n + (L_x + L_t + L_z + N)\tilde{u}_n - g\}\mathrm{d}s, \tag{2c}$$

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^z \lambda_4\{L_s u_n + (L_x + L_y + L_t + N)\tilde{u}_n - g\}\mathrm{d}s. \tag{2d}$$

where λ_1, λ_2, λ_3 and λ_4 are general Lagrange multipliers [18], which can be identified optimally via the variational theory [5,18], and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. By this method, we determine first the Lagrange multipliers λ_i $(i = 1,2,3,4)$ which will be identified optimally. The successive approximations u_{n+1}, $n \geqslant 0$, of the solution u will be readily obtained by suitable choice of trial function u_0. Consequently, the solution is given as

$$u(x,y,z,t) = \lim_{n\to\infty} u_n(x,y,z,t). \tag{3}$$

Applications of the VIM to various nonlinear problems can be found in details in Refs. [23–33]. In this paper, we will apply the VIM to the following problems to illustrate the strength of the method.

2.1. Parabolic-like equations

In this section, we consider the parabolic like equation in three dimensions which can be written in the form:

$$u_t + f_1(x,y,z)u_{xx} + f_2(x,y,z)u_{yy} + f_3(x,y,z)u_{zz} = 0, \tag{4}$$

with the initial condition

$$u(x,y,z,0) = f_4(x,y,z). \tag{5}$$

According to the VIM, we construct a correction functional for Eq. (4) in the form

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t \lambda\{(u_n)_s + f_1(x,y,z)(\tilde{u}_n)_{xx} + f_2(x,y,z)(\tilde{u}_n)_{yy} + f_3(x,y,z)(\tilde{u}_n)_{zz}\}\mathrm{d}s, \tag{6}$$

where $n \geqslant 0$ and $u_0(x,y,z,t) = u(x,y,z,0)$.

The VIM solution $u(x,y,z,t)$ can be expressed as

$$u(x,y,z,t) = \lim_{n\to\infty} u_n(x,y,z,t), \tag{7}$$

where $u_n(x,y,z,t)$ will be determined in a recursive manner.