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A direct method for the numerical computation of bifurcation points underlying symmetries

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Abstract

A direct method for the numerical computation of pitchfork bifurcation points under certain symmetry conditions is presented. We will be interested in computing the critical parameter. The direct method presented produces a larger system of full rank and hence solvable. The presence of symmetry will be of help since it will reduce the amount of work needed. Numerical experimentation will be done to demonstrate the efficiency of the suggested approach.

1. Introduction

Parameter dependent problems are common in engineering and applied sciences. They can be in different shapes and formats like two point boundary value problems, partial differential equations and integral equations. We will be interested in problems in the presence of parameters, one or two parameters. In particular, we will consider singular problems which will give rise to generalized turning points like quadratic and cubic turning points and bifurcation points. Recently enough work was done in terms of symmetry breaking bifurcation points that arise in applied sciences and engineering. For example, the buckling instability of the Euler elastics in civil engineering, El-Naschie [8], bifurcation from nontrivial static solutions of the nonlinear evolution equation, Li and Yang [12], in delay differential and difference equations, respectively, Peng [13] and Xiong [21], in the planar pendulum when parametrically excited by a periodic vertical force, Bishop et al. [5] and Sofroniou and Bishop [16] and finally in the general Helmholtz–Duffing oscillator, Cao et al. [6]. Of special interest is the critical parameter which is of significant physical meaning, see Domokosa et al. [7], Langaa et al. [11] and Varela et al. [18].

We will consider a parameter dependent problem that has the form

$$G(y, \lambda) = 0, \tag{1.1}$$

where G is a C^2-function which maps $\mathscr{R}^n \times \mathscr{R}$ into $\mathscr{R}^n$. As mentioned above, this can be obtained from a discretization of two point boundary value problems, partial differential equations or integral equations. At the critical parameter(s),

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the Jacobian of the system with respect to y given in (1.1) will be singular and has a drop in rank by one while the full Jacobian maintains rank n. As a result any conventional numerical method will suffer from convergence difficulties or in some cases diverges. This gives rise to what is called generalized turning and bifurcation points. The numerical computation of such points require some special treatment. Here we will consider a direct method for the numerical computation of a bifurcation problem that satisfies certain symmetry conditions; namely, a symmetry breaking bifurcation point or a pitchfork bifurcation point. For more on the computation of turning and bifurcation points (or more generally of singular points) of such systems, various types of (arclength parameterization, extended, augmented, defining,...) systems of equations were used, (see [1–3,10,17,20]).

The point (y_0, λ_0) is called a regular point of $G(y, \lambda) = 0$ if $G_y(y_0, \lambda_0)$ is nonsingular. It is called a simple singular point of $G(y, \lambda) = 0$ if $G_y(y_0, \lambda_0)$ is singular; $G_y^0 = G_y(y_0, \lambda_0)$ and $G_y^{0\mathrm{T}}$ each has one dimensional null space spanned by ϕ_0 and ψ_0, respectively. Whether or not bifurcation occurs at a singular point depends on the type of singularity, which is described by (in)equalities involving the derivatives of $G(y, \lambda)$ at (y_0, λ_0). Typically, turning points, transcritical and pitchfork bifurcation occur. To compute such points, we use an extended system with minimal number of added equations that characterizes the singular manifold while (1.1) characterizes the solution manifold. It is the intersection between the manifolds that we are after which will be isolated point(s), more on this system will be given in the next section (see also [1,2]).

The outline of this paper will be as follows. Some assumptions and definitions are given in Section 2. In Section 3, we will present the relation between symmetry breaking bifurcation points and pitchfork bifurcation, and details on the extended system will also be presented. In this section, we will also present the main theorem of this paper which shows that a symmetry breaking bifurcation point is an isolated solution of the proposed extended system. Numerical details and examples from applied sciences are given in the final section.

2. Characterization of the singularity

Recall the operator equation given by (1.1) of the form

$$G(y, \lambda) = 0 \tag{2.1}$$

with $G : R^n \times R \rightarrow R^n$. We will study the singularities of G as a function of y where λ enter generically and without any special dependencies. For that reason, assume that the matrix G_y has rank deficiency and it drops by one. This means that at some (y_0, λ_0) (with $G_y^0 \equiv G_y(y_0, \lambda_0)$), we have $G(y_0, \lambda_0) = 0$. Assume that $N(G_y^0) = \mathrm{span}\{\phi_0\}$, $R(G_y^0) = \{x \epsilon R^n; \psi_0^{\mathrm{T}} x = 0\}$ and $\psi_0^{\mathrm{T}} G_\lambda \neq 0$. These conditions implies that (y_0, λ_0) is a simple turning point of G with respect to the parameter λ. In order to characterize and hence to compute such points, choose $r \epsilon R^n$ and a linear mapping $t : R^n \rightarrow R$ such that the matrix

$$A = \begin{bmatrix} G_y & r \\ t & 0 \end{bmatrix} \tag{2.2}$$

vhas a bounded inverse in a domain $D \subset R^n$. Such r and t exist, for example if one chooses $r = G_\lambda$ and t a linear functional that is linearly independent on the null space of G_y^0 then A will be nonsingular. Now consider the solution of the system

$$G = -rg; \quad ty = s, \tag{2.3}$$

with $s \epsilon R$ and $g = g(s, \lambda)$ an intermediate scalar function. Differentiating (2.3) with respect to s, we obtain

$$G_x y_s = -r g_s; \quad t y_s = 1 \tag{2.4}$$

or in matrix format

$$A \begin{bmatrix} y_s \\ g_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.5}$$

It can easily be seen that G_y is deficient at some s if g_s vanishes. Hence y_s will then be in the null space of G_y. The fold points of the intermediate equation are the solutions of the 2×2 system

$$\begin{aligned} g(s, \lambda) &= 0 \\ g_s(s, \lambda) &= 0. \end{aligned} \tag{2.6}$$

Notice that this involves $g(s, \lambda)$ implicitly. This fact makes such process not very attractive and hence this 2×2 system will not be the one used to compute the critical point. For example instead of solving (2.6) for the simple turning point, we find y_s and g_s directly as a function of (y, λ); that is, with $y_s \equiv v \epsilon R^n$, $g_s \equiv g \epsilon R$ and $u^{\mathrm{T}} \epsilon R^n$ solve the systems