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On leafwise conformal diffeomorphisms

Kamil Niedziałomski

Department of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland

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ABSTRACT

For every diffeomorphism $\varphi : M \to N$ between 3-dimensional Riemannian manifolds M and N, there are locally two 2-dimensional distributions $D_\pm$ such that φ is conformal on both of them. We state necessary and sufficient conditions for a distribution to be one of $D_\pm$. These are algebraic conditions expressed in terms of the self-adjoint and positive definite operator induced from φ_*. We investigate the integrability condition of D_+ and D_-. We also show that it is possible to choose coordinate systems in which leafwise conformal diffeomorphism is holomorphic on leaves.

1. Introduction

Let $\varphi : M \to N$ be a diffeomorphism between 3-dimensional Riemannian manifolds (M, g) and (N, h). Fix $x \in M$ and let $(\varphi_{*x})^* : T_{\varphi(x)}N \to T_xM$ denotes the operator adjoint to $\varphi_{*x} : T_xM \to T_{\varphi(x)}N$. Then $S_x = (\varphi_{*x})^*\varphi_{*x}$ is a self–adjoint and positive definite operator. Let $0 < \lambda_1(x) \le \lambda_2(x) \le \lambda_3(x)$ be the eigenvalues of S_x.

Preimage $E(x) = \varphi_{*x}^{-1}(\mathbb{S}^2)$ of the unit sphere is an ellipsoid with principal semi-axes $1/\sqrt{\lambda_i(x)}$, $i = 1, 2, 3$. Therefore, if the eigenvalues $\lambda_i(x)$, $i = 1, 2, 3$, are distinct, there are two 2-dimensional subspaces $D_+(x)$ and $D_-(x)$ of T_xM intersecting $E(x)$ along spheres. Thus, locally we get two smooth distributions, D_+ and D_-. By the definition of $D_\pm$ we see that φ is conformal on each of them (see Lemma 1).

In this article, we describe D_+ and D_- and study the problem of integrability of these distributions. We show that integrability of one of the distributions $D_\pm$ does not imply integrability of the other one.

Conformality of diffeomorphisms on distributions of codimension one was studied by Tanno in [1,2]. However, the majority of results in [1,2] are obtained under the assumption that a given diffeomorphism φ maps vectors normal to a distribution D to vectors normal to the image $\varphi_*(D)$. Therefore φ cannot have distinct eigenvalues. Moreover, in [3] the author showed that under some assumptions on a diffeomorphism φ and the dimension of M, there are no distributions of 'small' codimension on which φ is conformal. In particular, assuming $\dim M > 3$ there are no codimension one foliations such that a diffoemorphism $\varphi : M \to N$, for which S has distinct eigenvalues, is conformal on the leaves.

The paper is organized as follows. In Section 2 we obtain preliminary results concerning some operators defined for 1-forms. Next, we state necessary and sufficient conditions for a diffeomorphism between 3-dimensional Riemannian manifolds to be conformal on a given distribution, that is we obtain conditions for a distribution to be one of $D_\pm$ (Theorem 4). Examples are given. In the following sections, we focus on the integrability condition of D_+ and D_- (Theorem 6, Propositions 7 and 8). The last part of this article is devoted to local descriptions of leafwise conformal diffeomorphism.

E-mail address: kamiln@math.uni.lodz.pl.

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We show that it is possible to choose appropriate coordinate systems in which given leafwise conformal diffeomorphism is holomorphic on leaves (Theorem 9).

2. Notations and preliminary results

Let (M, g), (N, h) be 3-dimensional oriented and connected Riemannian manifolds and let $\varphi : (M, g) \to (N, h)$ be a diffeomorphism. We say that φ is *leafwise conformal* if there exists a 2-dimensional foliation $\mathcal{F}$ on M such that $\varphi : L \to \varphi(L)$ is conformal for every leaf $L \in \mathcal{F}$. In that case we also say that φ is $\mathcal{F}$*–conformal*. φ is *locally leafwise conformal* if every point $x \in M$ has a neighborhood U such that $\varphi : U \to \varphi(U)$ is leafwise conformal.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the operator $S = (\varphi_*)^*\varphi_* : TM \to TM$ and ξ_1, ξ_2, ξ_3 be the corresponding unit eigenvectors. Assume $\lambda_1 < \lambda_2 < \lambda_3$. Let η_1, η_2, η_3 be the basis dual to ξ_1, ξ_2, ξ_3. Locally, we may choose the above bases to be smooth. Define

$$\omega_\pm = \frac{\sqrt{\lambda_2 - \lambda_1}}{\sqrt{\lambda_3 - \lambda_1}}\eta_1 \pm \frac{\sqrt{\lambda_3 - \lambda_2}}{\sqrt{\lambda_3 - \lambda_1}}\eta_3. \tag{1}$$

Consider the distributions $D_\pm = \ker \omega_\pm$. We have

Lemma 1. *Diffeomorphism φ is (locally) conformal on a 2-dimensional distribution D if and only if $D = D_+$ or $D = D_-$ (locally). Moreover, the coefficient of conformality is λ_2.*

Proof. It is easy to check that φ is conformal on D_+ and D_- with coefficient of conformality λ_2. Suppose there exists a distribution D such that φ is conformal on D. Fix $x \in M$ and consider the set $E(x) = d\varphi^{-1}(x)(\mathbb{S}^2)$, where $\mathbb{S}^2 \subset T_{\varphi(x)}N$ is the unit sphere. Then $E(x)$ is an ellipsoid with principal semi-axes $1/\sqrt{\lambda_i(x)}$, $i = 1, 2, 3$. The subspaces $D_+(x)$ and $D_-(x)$ intersect $E(x)$ along circles and these are the only subspaces with this property, see [4] or [3]. Thus, by conformality of φ on D we get that $D(x) = D_+(x)$ or $D(x) = D_-(x)$. Since M is connected, D is smooth and $D_+(x) \neq D_-(x)$ for all $x \in M$, we obtain $D = D_+$ or $D = D_-$ (locally). □

Let $x \in M$, $p = 0, 1, 2, 3$ and $* : \Lambda^p T_x^*M \to \Lambda^{3-p}T_x^*M$ be the Hodge operator. Let $\iota(\omega)\eta = \omega \wedge \eta$ for $\omega, \eta \in \Lambda^p T_x^*M$. For $\omega, \eta \in T_x^*M$ define $(\omega \odot \eta)_x : T_x^*M \to T_x^*M$ by

$$(\omega \odot \eta)_x \alpha = \langle \omega, \alpha \rangle \eta + \langle \eta, \alpha \rangle \omega, \quad \alpha \in T_x^*M,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in T_x^*M induced from Riemannian metric g. Moreover for $\theta \in [0, 2\pi)$ and $\omega \in T_x^*M$, $|\omega| = 1$, put

$$\mathrm{Rot}_x(\theta, \omega) = \mathrm{Id}_{T_x^*M} + \sin\theta(*\iota(\omega)) + (1 - \cos\theta)(*\iota(\omega))^2 : T_x^*M \to T_x^*M.$$

Then $\mathrm{Rot}_x(\theta, \omega)$ is an operator of rotation around ω of an angle θ, for details see [5]. For simplicity, we will write $\mathrm{Rot}_x(\omega)$ instead of $\mathrm{Rot}_x(\pi/2, \omega)$.

Lemma 2. *Let $0 \le \theta, \theta_1, \theta_2 < 2\pi$, $\omega, \eta \in T_x^*M$ and $|\omega| = 1$. The operator $\mathrm{Rot}_x(\theta, \omega)$ has the following properties*

(1) $\mathrm{Rot}_x(\theta_1, \omega) \circ \mathrm{Rot}_x(\theta_2, \omega) = \mathrm{Rot}_x(\theta_1 + \theta_2 \bmod 2\pi, \omega)$.
(2) *If $\langle \omega, \eta \rangle = 0$ then $\langle \omega, \mathrm{Rot}_x(\theta, \omega)\eta \rangle = 0$.*
(3) *If $\langle \omega, \eta \rangle = 0$ then $\langle \mathrm{Rot}_x(\omega)\eta, \eta \rangle = 0$ and $\eta - \mathrm{Rot}_x(\omega)\eta = \sqrt{2}\mathrm{Rot}_x(-\frac{\pi}{4}, \omega)\eta$.*

Proof. Easy computations left to the reader. □

The operator $S_x : T_xM \to T_xM$ can be considered as an operator $S_x : T_x^*M \to T_x^*M$ by the rule $(S_x\eta)X = \eta(SX)$, $X \in T_xM$. Then S is a self–adjoint and positive definite operator with eigenvalues λ_i and corresponding eigenvectors η_i, $i = 1, 2, 3$. Let $[T_1, T_2] = T_1T_2 - T_2T_1 : T_x^*M \to T_x^*M$ be the commutator of operators $T_1, T_2 : T_x^*M \to T_x^*M$. We define

$$B_x(\omega) = [S_x, *\iota(\omega)] : T_x^*M \to T_x^*M, \tag{2}$$

$$A_x(\omega) = [S_x, \mathrm{Rot}_x(\omega)] : T_x^*M \to T_x^*M. \tag{3}$$

We have a technical result

Lemma 3. *Let $\omega \in T_x^*M$, $|\omega| = 1$. Then there exist $\eta, \sigma \in T_x^*M$ such that ω, η, σ are orthogonal and*

$$S_x\eta = \frac{1}{|\eta|^2}\eta + \langle S_x\omega, \eta \rangle \omega, \qquad S_x\sigma = \frac{1}{|\sigma|^2}\sigma + \langle S_x\omega, \sigma \rangle \omega. \tag{4}$$

Proof. Let $\omega = \sum_i a_i\eta_i$. If $\omega = \eta_i$ for some $i = 1, 2, 3$, then it suffices to put $\eta = (1/\sqrt[3]{\lambda_j})\eta_j$ and $\sigma = (1/\sqrt[3]{\lambda_k})\eta_k$, where (i, j, k) is a permutation of the set $\{1, 2, 3\}$.

Suppose now $\omega \neq \eta_i$ for all $i = 1, 2, 3$. Let $C > 0$ be such that $\sum_i a_i^2/(\lambda_i - C) = 0$ and put $\eta = \sum_i (a_i/(\lambda_i - C))\eta_i$. Then $\langle \omega, \eta \rangle = 0$ and $S_x\eta = C\eta + \omega$. It suffices to multiply η by $1/\sqrt{C}|\eta|$. Let $\sigma = \mathrm{Rot}_x(\omega)\eta$. By Lemma 2 ω, η, σ are orthogonal. Moreover, $\langle S_x\sigma, \eta \rangle = 0$ and $\langle S_x\sigma, \sigma \rangle > 0$, thus multiplying σ by an appropriate factor we get $S_x\sigma = \frac{1}{|\sigma|^2}\sigma + \langle S\omega, \sigma \rangle \omega$. □