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Deforming the Lie superalgebra of contact vector fields on $S^{1|2}$

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We classify the nontrivial deformations of the standard embedding of the Lie superalgebra K(2) of contact vector fields on the (1,2)-dimensional supercircle into the Lie superalgebra of superpseudodifferential operators on the supercircle. This approach leads to the deformations of the central charge induced on K(2) by the canonical central extension of $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$.

1. Introduction

The classical deformation theory of Lie algebras and modules over Lie algebras traditionally deals with one parameter deformations [1–3]. It is, however, natural to consider, as in other deformation theories, multi-parameter deformations. The viewpoint of multi-parameter deformations of Lie algebra over commutative algebras was initiated by Fialowski (see [4]) and the construction of miniversal deformations was given by Fialowski and Fuks in [5]. Similar constructions of deformations of homomorphisms of infinite dimensional Lie algebras are considered by Ovsienko and Roger (see [6,7]). The Lie algebra $\mathrm{Vect}(S^1)$ of vector fields in the circle S^1 is rigid. In [6] (resp. [7]), the deformations of the natural embedding of $\mathrm{Vect}(S^1)$ inside the Poisson algebra of the Laurent series on $\dot{T}^*(S^1)$ (resp. the Lie algebra of pseudodifferential operators) have been classified. In [8]the multiparameter deformations of the Lie derivative action of the Lie algebra $\mathrm{Vect}(\mathbb{R}^n)$ of vector fields on $\mathbb{R}^n$ on the space of symmetric tensor fields were considered. The deformations of the module of differential forms on $\mathbb{R}^n$ was studied in [9].

The first step of any approach to the deformation theory consists of the study of infinitesimal deformations. Given a Lie algebra (or superalgebra) $\mathfrak{g}$ and a $\mathfrak{g}$-module V, the infinitesimal deformation is defined, up to equivalence, by the cohomology classes $c_1, \ldots, c_n$ in $H^1(\mathfrak{g}, \mathrm{End}(V))$.

Consider the Lie algebra $\mathrm{Vect}(M)$ of vector fields on a manifold M and the space of differential operators on M. The adjoint action of $\mathrm{Vect}(M)$ on $\mathcal{DO}(M)$ preserves the degree filtration $\mathcal{DO}(M)^k$ of $\mathcal{DO}(M)$ and induces a $\mathrm{Vect}(M)$-module on the space of symbols $\mathcal{S}^k = \mathcal{DO}(M)^k/\mathcal{DO}(M)^{k-1}$ which is the space of the symmetric tensor fields module of degree k. The infinitesimal deformation of the action by Lie derivatives of $\mathrm{Vect}(M)$ on $\mathcal{S}^*$ is classified by $H^1(\mathrm{Vect}(M), D(\mathcal{S}^*, \mathcal{S}^*))$ where $D(\mathcal{S}^*, \mathcal{S}^*) \subset \mathrm{End}(\mathcal{S}^*)$ is the space of differential operators between tensor fields. The space $H^1(\mathrm{Vect}(M), D(\mathcal{S}^*, \mathcal{S}^*))$ was calculated, for an arbitrary manifold M of dimension ≥ 2 by Lecomte and Ovsienko (see [10]). For the one dimensional case ($M = \mathbb{R}$ or $M = S^1$), the modules $\mathcal{S}^k$ are identified with modules of tensor densities $\mathcal{F}_k$ of degree k of $\mathrm{Vect}(M)$. The first cohomology space $H^1(\mathrm{Vect}(M), D(\mathcal{F}_\lambda, \mathcal{F}_\mu))$ was computed by Feigin and Fuks [11] where $\lambda, \mu \in \mathbb{C}$.

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For the vector fields Lie superalgebra case, the first examination appears in the end of the seminal paper by Cohen, Manin, and Zagier (see [12]) where the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on $\mathbb{R}^{1|1}$ was considered and a conformal symbol is computed for the space of differential operators $\mathcal{DO}(\mathbb{R}^{1|1})$ on $\mathbb{R}^{1|1}$. In a series of recent papers, the cohomology of the Lie superalgebra $\mathcal{K}(n)$ of contact vector fields on $S^{1|n}$ (or $\mathbb{R}^{1|n}$) is considered and multiparameter deformation has been studied: The pseudodifferential operator module-valued first cohomology groups of $\mathcal{K}(1)$ for the adjoint action, are computed by the first author and Ben Fraj in [13], the multiparameter deformation of the associated infinitesimal deformation is studied by Ben Fraj and Omri (see [14]). It is shown there that each formal deformation is equivalent to an infinitesimal one, a less rich situation than that of the $\mathrm{Vect}(S^1)$ case computed by Ovsienko and Roger see [7] for details.

In [15], the first author, Ben Fraj and Omri computed the first cohomology space $H^1(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$. This space is ten dimensional.

In this paper, we will compute the integrability conditions of infinitesimal deformations of the standard embedding of the Lie superalgebra $\mathcal{K}(2)$ of contact vector fields on the supercircle $S^{1|2}$ inside the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$ of superpseudodifferential operators on $S^{1|2}$. The infinitesimal deformations of this embedding are classified by $H^1(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$. The obstructions for the integrability of infinitesimal deformations lie in $H^2(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$. Our goal is to study these obstructions. We will follow mostly the same strategy as in [7].

2. The main definitions

2.1. Super-pseudo-differential operators on super-circles

We first recall the definition of $\mathcal{S}\Psi\mathcal{DO}(S^{1|n})$, $n \in \mathbb{N}$ (cf. [16,17]). The super-circle $S^{1|n}$ is the superextension of the circle S^1 with local coordinates $(\varphi, \theta_1, \ldots, \theta_n)$, where $x = e^{i\varphi} \in S^1$ and $(\theta_1, \ldots, \theta_n)$ are odd variables satisfying the relations $\theta_i\theta_j = -\theta_j\theta_i$, $\forall 1 \le i, j \le n$. For a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we denote by $p(a)$ the parity of $a \in V$. A C^∞-function on $S^{1|n}$ has the form

$$F = \sum_{s=0}^{n} \sum_{1\le i_1<i_2<\cdots<i_s\le n} f_{i_1 i_2 \ldots i_s}(x)\theta_{i_1}\cdots\theta_{i_s}, \tag{2.1}$$

where $x = e^{i\varphi}$ and $f_{i_1 i_2 \ldots i_s} \in C^\infty(S^1)$. For $V = C^\infty(S^{1|n})$, we let $p(x) = 0$ and $p(\theta_i) = 1$.

The super-space of the super-commutative algebra of super-pseudo-differential symbols on $S^{1|n}$ with its natural multiplication is spanned by the series

$$\mathcal{SP}(n) = \left\{ A = \sum_{k=-M}^{\infty} \sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_n)} a_{k,\epsilon}(x,\theta)\xi^{-k}\bar{\theta}_1^{\epsilon_1}\cdots\bar{\theta}_n^{\epsilon_n} | a_{k,\epsilon} \in C^\infty(S^{1|n});\ \epsilon_i = 0, 1;\ M \in \mathbb{N} \right\}, \tag{2.2}$$

where ξ corresponds to ∂_x and $\bar{\theta}_i$ corresponds to ∂_{θ_i} $(p(\bar{\theta}_i) = 1)$. The space $\mathcal{SP}(n)$ has a structure of the Poisson Lie superalgebra given by the following bracket:

$$\{A, B\} = \frac{\partial(A)}{\partial\xi}\frac{\partial(B)}{\partial x} - \frac{\partial(A)}{\partial x}\frac{\partial(B)}{\partial\xi} - (-1)^{p(A)}\sum_{i=1}^{n}\left(\frac{\partial(A)}{\partial\theta_i}\frac{\partial(B)}{\partial\bar{\theta}_i} + \frac{\partial(A)}{\partial\bar{\theta}_i}\frac{\partial(B)}{\partial\theta_i}\right).$$

The associative super-algebra of super-pseudo-differential operators $\mathcal{S}\Psi\mathcal{DO}(S^{1|n})$ on $S^{1|n}$ has the same underlying vector space as $\mathcal{SP}(n)$, but the multiplication is now defined by the following rule:

$$A \circ B = \sum_{\alpha\ge 0, \nu_i=0,1} \frac{(-1)^{p(A)+1}}{\alpha!}(\partial_\xi^\alpha \partial_{\bar{\theta}_i}^{\nu_i} A)(\partial_x^\alpha \partial_{\theta_i}^{\nu_i} B).$$

This composition rule induces the super-commutator defined by:

$$[A, B] = A \circ B - (-1)^{p(A)p(B)} B \circ A.$$

One can construct the contraction of the Lie super-algebra $\mathcal{S}\Psi\mathcal{DO}(S^{1|n})$ to the Poisson algebra $\mathcal{SP}(n)$. Let us consider the super-commutative super-algebra Λ with generators $(\theta_1, \ldots, \theta_n; \bar{\theta}_1 \ldots, \bar{\theta}_n)$. Then $\mathcal{SP}(n) = \mathcal{P} \otimes \Lambda$, where $\mathcal{P}$ is the space of symbols of ordinary pseudo-differential operators $\Psi\mathcal{DO}(S^1)$. Let us introduce the linear isomorphisms

$$\Phi_h : \mathcal{P} \longrightarrow \mathcal{P}$$

defined by

$$\Phi_h(a(x)\xi^l) = a(x)h^l\xi^l, \quad \text{where } h \in]0, 1].$$

The new multiplication on $\mathcal{P}$ is defined by

$$A \circ_h B = \Phi_h^{-1}(\Phi_h(A) \circ \Phi_h(B)).$$

We also modify the Leibniz rule in the odd variables by letting:

$$\bar{\theta}_i \cdot \theta_j = h\delta_{i,j} - \theta_j\bar{\theta}_i.$$