Journal of Geometry and Physics 84 (2014) 73–86

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On rational Frobenius manifolds of rank three with symmetries

Alexey Basalaev [a,b,*], Atsushi Takahashi [c,1]

[a] National Research University Higher School of Economics, Vavilova 7, 117312 Moscow, Russia
[b] Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
[c] Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan

ARTICLE INFO

Article history:
Received 4 March 2014
Received in revised form 26 May 2014
Accepted 27 May 2014
Available online 4 June 2014

MSC:
53D45
32S30
11G05

Keywords:
Frobenius manifolds
Singularity theory
Modular forms
Elliptic curves

ABSTRACT

We study Frobenius manifolds of rank three and dimension one that are related to submanifolds of certain Frobenius manifolds arising in mirror symmetry of elliptic orbifolds. We classify such Frobenius manifolds that are defined over an arbitrary field $\mathbb{K} \subset \mathbb{C}$ via the theory of modular forms. By an arithmetic property of an elliptic curve $\mathcal{E}_\tau$ defined over $\mathbb{K}$ associated to such a Frobenius manifold, it is proved that there are only two such Frobenius manifolds defined over $\mathbb{C}$ satisfying a certain symmetry assumption and thirteen Frobenius manifolds defined over $\mathbb{Q}$ satisfying a weak symmetry assumption on the potential.

0. Introduction

The notion of a Frobenius manifold was introduced by Boris Dubrovin in the 1990s (cf. [1]) as the mathematical axiomatization of a 2D topological conformal field theory. A special class of Frobenius manifolds is given by certain structures on the base space of the universal unfolding of a hypersurface singularity. These structures were introduced in the early 1980s by Kyoji Saito (cf. [2] for an introduction to this theory) and called at that time Saito's flat structures.

Actually, Saito found a richer structure than his flat structure, consisting of the filtered de Rham cohomology with the Gauß–Manin connection, higher residue pairings and a primitive form [3]. Unlike the general setting of a Frobenius manifold it has much more geometric data coming naturally from singularity theory. It is also now generalized as a so-called non-commutative Hodge theory by [4] which will be a necessary tool to understand the classical mirror symmetry (isomorphism of Frobenius manifolds) via a Kontsevich's homological mirror symmetry.

It is a very important problem to study some arithmetic aspect of a Saito structure with a geometric origin such as singularity theory. However, it is quite difficult at this moment. Therefore we start our consideration from the larger setting of Frobenius manifolds.

* Corresponding author at: Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany. Tel.: +7 9035904648, +49 511762.
E-mail addresses: aabasalaev@edu.hse.ru, basalaev@math.uni-hannover.de (A. Basalaev), takahashi@math.sci.osaka-u.ac.jp (A. Takahashi).
1 Tel.: +81 6 6850 5326.

http://dx.doi.org/10.1016/j.geomphys.2014.05.030
0393-0440/

Namely we define particular $GL(2,\mathbb{C})$-action on the Frobenius manifolds of rank 3 and dimension 1. This action corresponds to the change of the primitive form of the simple elliptic singularities. More precisely, we shall define the Frobenius manifold $M^{(\tau_0,\omega_0)}$ of rank three and dimension one obtained by acting with a certain element $A^{(\tau_0,\omega_0)} \in GL(2,\mathbb{C})$ depending on $\tau_0 \in \mathbb{H}$, $\omega_0 \in \mathbb{C}\setminus\{0\}$ (see Section 2.5) on the "basic" Frobenius manifold M^∞ (see Proposition 2.9 for the definition of M^∞). The Frobenius manifold M^∞ itself is connected to the Gromov–Witten Frobenius structures of the orbifold projective lines $\mathbb{P}^1_{2,2,2,2}$, $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$. These orbifold projective lines provide the Calabi–Yau/Landau–Ginzburg mirror symmetry for simple elliptic singularities (see [5]), which involves choosing the primitive form at the so-called *large complex structure limit* (LCSL for brevity). Therefore we can consider M^∞ as corresponding to the primitive form choice at the LCSL.

The general context of the global mirror symmetry requires the existence of the so-called orbifolded Landau–Ginzburg A-model that is the Frobenius manifold, associated to the pair—singularity and a symmetry group of it. The systematic approach to this problem was given in [6] and is now called FJRW-theory. However it appears to be very hard to compute.

Looking for the Frobenius manifold that could potentially serve an orbifolded Landau–Ginzburg A-model it is natural to assume it to have some special properties. Namely to be defined over $\mathbb{Q}$ and have some "symmetries". By the global mirror symmetry assumption the orbifolded Landau–Ginzburg A-model should also correspond to some primitive form choice. This motivates our classification of the Frobenius manifolds $M^{(\tau_0,\omega_0)}$ defined over the field $\mathbb{K}\subset\mathbb{C}$ and also having "symmetries".

Results

Let $\mathbb{K}\subset\mathbb{C}$ be a field. We say that a Frobenius manifold M is *defined over* $\mathbb{K}$ if there exist flat coordinates $t_1,\ldots,t_\mu$ in which the Frobenius potential of M belongs to $\mathbb{K}\{t_1,\ldots,t_\mu\}$ and is defined at the point $t_1=\cdots=t_\mu=0$.

We associate the elliptic curve $\mathcal{E}_{\tau_0}$ with the modulus τ_0 with $M^{(\tau_0,\omega_0)}$. The first theorem of this paper states several criteria of the Frobenius manifold $M^{(\tau_0,\omega_0)}$ to be defined over $\mathbb{K}$. The criteria are given in terms of the values of the modular forms at the point $\tau_0\in\mathbb{H}$.

In what follows we translate some properties of the elliptic curve $\mathcal{E}_{\tau_0}$ into special properties of the Frobenius manifold $M^{(\tau_0,\omega_0)}$. Considering the $SL(2,\mathbb{R})$-action on $M^{(\tau_0,\omega_0)}$ we define the property of the Frobenius manifold $M^{(\tau_0,\omega_0)}$ to be "symmetric" and "weakly symmetric". Namely we call $M^{(\tau_0,\omega_0)}$ symmetric if its potential is preserved by the action of some $A\in SL(2,\mathbb{R})$ and weakly symmetric if its potential is rescaled by the action of A.

In the second theorem of this paper we show that the Frobenius manifold $M^{(\tau_0,\omega_0)}$ has a "symmetry" if and only if τ_0 is in the $SL(2,\mathbb{Z})$ orbit of $\sqrt{-1}$ or $\exp(2\pi\sqrt{-1}/3)$ and the Frobenius manifold $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ has a "weak symmetry" if and only if $\mathcal{E}_{\tau_0}$ is isomorphic to one of 13 elliptic curves listed in Corollary 4.3.

Organization of the paper

After recalling some basic definitions and terminologies in Section 1, we shall study a rational structure on $M^{(\tau_0,\omega_0)}$. The $GL(2,\mathbb{C})$-action and in particular $A^{(\tau_0,\omega_0)}$-action are defined in Section 2. In Section 3 we shall prove the first theorem of this paper and also give two natural examples of $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$. Section 4 is devoted to the study of the symmetries of $M^{(\tau_0,\omega_0)}$. It contains the second theorem of this paper. Finally, some useful data are given in the Appendix.

1. Preliminaries

1.1. Frobenius manifolds

We give some basic properties of a Frobenius manifold [1]. Let us recall the equivalent definition taken from Saito–Takahashi [2].

Definition. Let $M=(M,\mathcal{O}_M)$ be a connected complex manifold of dimension μ whose holomorphic tangent sheaf and cotangent sheaf are denoted by $\mathcal{T}_M$ and Ω^1_M respectively and let d be a complex number.

A *Frobenius structure of rank μ and dimension d on M* is a tuple $(\eta,\circ,e,E)$, where η is a non-degenerate $\mathcal{O}_M$-symmetric bilinear form on $\mathcal{T}_M$, $\circ$ is an $\mathcal{O}_M$-bilinear product on $\mathcal{T}_M$, defining an associative and commutative $\mathcal{O}_M$-algebra structure with a unit e, and E is a holomorphic vector field on M, called the Euler vector field, which are subject to the following axioms:

1. The product $\circ$ is self-adjoint with respect to η: that is,
$$\eta(\delta\circ\delta',\delta'')=\eta(\delta,\delta'\circ\delta''),\quad \delta,\delta',\delta''\in\mathcal{T}_M.$$
2. The Levi–Civita connection $\nabla:\mathcal{T}_M\otimes_{\mathcal{O}_M}\mathcal{T}_M\to\mathcal{T}_M$ with respect to η is flat: that is,
$$[\nabla_\delta,\nabla_{\delta'}]=\nabla_{[\delta,\delta']},\quad \delta,\delta'\in\mathcal{T}_M.$$
3. The tensor $C:\mathcal{T}_M\otimes_{\mathcal{O}_M}\mathcal{T}_M\to\mathcal{T}_M$ defined by $C_\delta\delta':=\delta\circ\delta'$, $(\delta,\delta'\in\mathcal{T}_M)$ is flat: that is,
$$\nabla C=0.$$
4. The unit element e of the $\circ$-algebra is a ∇-flat holomorphic vector field: that is,
$$\nabla e=0.$$