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A new method for constructing soliton solutions to differential-difference equation with symbolic computation

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Abstract

With the aid of the symbolic computation, we present a new method to find explicit exact solutions to nonlinear differential-difference equation. We successfully solve a lattice equation introduced by Wadati [Prog Theor Phys 1976;59 (Suppl.):36–63], and obtain some new soliton solutions.

1. Introduction

Discrete solitons in nonlinear lattices have been the focus of considerable attention in various branches of science. As is well-known, there are many physically interesting problems such as charge fluctuations in net-work, ladder type electric circuits, phenomena in crystals, molecular chains, and so on. Many of them can be modelled by nonlinear differential-difference equation(s) (DDE(s)). It makes sense to research for solving DDE(s).

Very recently, Baldwin et al. [1] presented an algorithm to find exact travelling wave solutions of NDDE in terms of tanh function and found kink-type solutions in many spatially discrete nonlinear models such as Ablowitz–Ladik lattice, Toda lattice, Volterra lattice, discrete mKdV lattice, Hybrid lattice. And later, Xie extended the method. However, Xie's method [2] is still unable to find solutions of polynomials in sech or csch forms.

In this paper, we introduce a new method and directly get rich soliton solutions for a lattice equation. In the next section, we will express the method.

2. Method and algorithm

Suppose the NDDE we discuss in this paper is in the following nonlinear polynomial form:

$$G(u_{n+p_1}(t), u_{n+p_2}(t), \ldots, u_{n+p_s}(t), u'_{n+p_1}(t), u'_{n+p_2}(t), \ldots, u'_{n+p_s}(t), \ldots, u^r_{n+p_1(t)}, u^r_{n+p_1(t)}, \ldots, u^r_{n+p_s(t)}) = 0, \tag{1}$$

where $u_n(t) = u(n,t)$ is a dependent variable; t is a continuous variable; n, $p_i \in Z$.

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To compute the travelling wave solutions to Eq. (1), we first set $u_n(t) = u(\xi_n)$, and

$$\xi_n = n \times d + ct + \xi_0. \tag{2}$$

Step 1: To assume the travelling wave solutions of Eq. (1) is in the following form:

$$u(\xi_n) = \sum_{i=-N}^{N} a_i \cosh(\omega_n)^i + \sum_{i=1}^{N} b_i \cosh(\omega_n)^{i-1}\sinh(\omega_n) + \sum_{i=-N}^{-1} c_i \cosh(\omega_n)^i \sinh(\omega_n), \tag{3}$$

with

$$\frac{d\omega_n}{d\xi_n} = \sinh(\omega_n), \tag{4}$$

where $\omega_n = \omega(\xi_n)$, $a_0, a_{\pm 1}, \ldots a_{\pm N}$, $b_1 \ldots b_N$, $c_{-1} \ldots c_{-N}$, and c are constants to be determined later, and N can be determined by balancing the highest degree linear term and nonlinear term of u_n.

Step 2: To derive the algebraic system.
Simple computation leads to the following identity:

$$\xi_{n+p_i} = (n + p_i)d + ct + \xi_0 = \xi_n + p_i \times d.$$

So,

$$u_{n+p_i} = \sum_{i=-N}^{N} a_i \cosh(\omega_{n+p_i})^i + \sum_{i=1}^{N} b_i \cosh(\omega_{n+p_i})^{i-1} \sinh(\omega_{n+p_i}) + \sum_{i=-N}^{-1} c_i \cosh(\omega_{n+p_i})^i \sinh(\omega_{n+p_i}). \tag{5}$$

Meanwhile, from Remark, we can derive

$$\cosh(\omega_{n+p_i}) = -\coth(\xi_{n+p_i}) = \frac{\cosh(\omega_n)\cosh(p_i d) - \sinh(p_i d)}{\cosh(p_i d) - \cosh(\omega_n)\sinh(p_i d)}, \tag{6}$$

and

$$\sinh(\omega_{n+p_i}) = -\mathrm{csch}(\xi_{n+p_i}) = \frac{\sinh(\omega_n)}{\cosh(p_i d) - \cosh(\omega_n)\sinh(p_i d)}. \tag{7}$$

Substituting (3)–(5) with (6), (7) into Eq. (1), then clearing the denominators, we obtain a finite series of $\sinh(\omega_n)^k \cosh(\omega_n)^i$ ($k = 0, 1$; $i = 0, 1 \ldots m$). Set the coefficients of $\sinh(\omega_n)^k \cosh(\omega_n)^i$ to zero, and we get a set of algebraic equations with respect to the unknown a_i, b_i, c_i, c.
Step 3: Solve the nonlinear over-determined algebraic equations, and we can obtain expressions of $u(\xi_n)$.

Remark

1. $\frac{d\sinh(\omega)}{d\omega} = \cosh(\omega)$, $\frac{d\cosh(w)}{d\omega} = \sinh(w)$, $\sinh(\omega)^2 = \cosh(\omega)^2 - 1$.
2. By using separation of variables method, if $\frac{d\omega}{d\xi} = \sinh(\omega)$, we can get $\sinh(\omega) = -\mathrm{csch}(\xi)$, and $\cosh(\omega) = -\coth(\xi)$;
3. $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$, and $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$.

3. Application of the method

The lattice equation can be expressed as

$$\frac{du_n(t)}{dt} = (\alpha + \beta u_n + \gamma u_n^2)(u_{n-1}(t) - u_{n+1}(t)), \tag{8}$$

where $\gamma \neq 0$. The equation contains Hybrid lattice equation, mKdV lattice equation, modified Volterra lattice equation:

(i) $(2+1)$dimensional Hybrid lattice equation [1]: $\frac{du_n(t)}{dt} = (1 + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1})$;
(ii) mKdV lattice equation [1,3]: $\frac{du_n(t)}{dt} = (\alpha - u_n^2)(u_{n-1} - u_{n+1})$;
(iii) modified Volterra lattice equation [4]: $\frac{du_n(t)}{dt} = u_n^2(u_{n-1} - u_{n+1})$.