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Gradient estimates and Liouville theorems for Dirac-harmonic maps

Qun Chen [a,b], Jürgen Jost [b,c], Linlin Sun [a,b,*]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
[b] Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany
[c] Department of Mathematics, Leipzig University, 04081 Leipzig, Germany

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ABSTRACT

In this paper, we derive gradient estimates for Dirac-harmonic maps from complete Riemannian spin manifolds into regular balls in Riemannian manifolds. With these estimates, we can prove Liouville theorems for Dirac-harmonic maps under curvature or energy conditions.

1. Introduction

Dirac-harmonic maps have been introduced in [1,2]. They couple a harmonic map type field with a spinor field [3]. This model originated in the supersymmetric σ-model of quantum field theory, the only difference being that in the supersymmetric σ-model the (anticommuting) spinor fields take values in a Grassmannian algebra, making the model supersymmetric, while in Dirac-harmonic maps, the spinors are commuting as in spin geometry, keeping the model within the category of the geometric calculus of variations.

Let us recall the terminology and setting for Dirac-harmonic maps. Let (M^m, g) be a Riemannian spin manifold of dimension $m \geq 2$ with a fixed spin structure, and ΣM the spinor bundle over M, on which we chose a Hermitian metric $\langle\cdot,\cdot\rangle$. The Levi-Civita connection ∇ on ΣM is compatible with $\langle\cdot,\cdot\rangle$. Let (N^n, h) be a Riemannian manifold of dimension n, Φ a map from M to N, and $\Phi^{-1}TN$ the pull-back bundle of TN by Φ. On the twisted bundle $\Sigma M \otimes \Phi^{-1}TN$ there is a metric (still denoted by $\langle\cdot,\cdot\rangle$) induced from the metrics on ΣM and $\Phi^{-1}TN$. There is also a connection, still denoted by ∇, on $\Sigma M \otimes \Phi^{-1}TN$ naturally induced from those on ΣM and $\Phi^{-1}TN$.

Locally, we can write a cross-section Ψ of $\Sigma M \otimes \Phi^{-1}TN$ as $\Psi = \psi^\alpha \otimes \theta_\alpha$, where $\{\psi^\alpha\}$ are local cross-sections of ΣM, $\{\theta_\alpha\}$ are local cross-sections of $\Phi^{-1}TN$. Here and in the sequel, we use the usual summation convention.

The *Dirac operator along the map* Φ is defined as

$$\begin{aligned} \not{D}\Psi &:= e_i \cdot \nabla_{e_i}\Psi \\ &= \not{\partial}\psi^\alpha \otimes \theta_\alpha + \psi^\alpha \otimes \nabla_{e_i}\theta_\alpha, \end{aligned}$$

* Corresponding author at: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail addresses: qunchen@whu.edu.cn (Q. Chen), jost@mis.mpg.de (J. Jost), sunll101@whu.edu.cn (L. Sun).

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where $\{e_i\}$ is a local orthonormal basis on M, $\partial\!\!\!/ := e_i \cdot \nabla_{e_i}$ is the usual Dirac operator on M and "$X\cdot$" stands for the Clifford multiplication by the vector field X on M.

Consider the functional

$$L(\Phi, \Psi) = \frac{1}{2}\int_M \left(\|d\Phi\|^2 + \langle \Psi, D\!\!\!\!/\,\Psi\rangle\right).$$

The critical points (Φ, Ψ) satisfy the Euler–Lagrange equations for $L(\Phi, \Psi)$ are (cf. [1])

$$\begin{cases} \tau(\Phi) = \frac{1}{2}\langle \psi^\alpha, e_i \cdot \psi^\beta\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i), \\ D\!\!\!\!/\,\Psi = 0, \end{cases} \tag{1.1}$$

where $R^N(X, Y) := [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}, \forall X, Y \in \Gamma(TN)$ stands for the curvature operator of N, and $\tau(\Phi) := (\nabla^{T^*M\otimes\Phi^{-1}TN}_{e_i} d\Phi)(e_i)$ is the tension field of Φ. Therefore, solutions of (1.1) are called *Dirac-harmonic maps from M to N*.

Dirac-harmonic maps have been investigated under various aspects, see the recent article [4] and the references therein. In [4], a maximum principle of Jäger–Kaul type [5] was established for Dirac-harmonic maps from compact Riemannian spin manifolds with mean convex boundaries and positive scalar curvatures into certain geodesic balls of the target manifolds, based on which a general existence and uniqueness theorem for boundary value problems was proved through the continuity method. Most recently, the space of Dirac-harmonic maps was analyzed by B. Ammann and N. Ginoux in [6] by using tools from index theory, and the existence of uncoupled solutions (i.e., Φ is a harmonic map) was proved.

Most of the previous works deal with Dirac-harmonic maps from compact manifolds. It is the main aim of the present paper to derive properties of Dirac-harmonic maps on complete noncompact manifolds M.

In the classical works of S.T. Yau [7] and others on harmonic functions on noncompact manifolds, the gradient estimate method plays a key role. On one hand, these estimates may directly give rise to Liouville type results; on the other hand, they may also lead to fundamental analytic properties such as Harnack inequalities, and furthermore, they are very useful for establishing existence results. This method has been extended to the case of harmonic maps. In [8], S.Y. Cheng established gradient estimates and derived the Liouville theorem for harmonic maps from a noncompact manifold M into a nonpositively curved manifold N. In [9] H.I. Choi proved a similar result for harmonic maps into a *regular ball*, namely, a geodesic ball $B_{y_0}(R)$ with radius R that lies within the cut locus of its center $y_0 \in N$ and satisfies $R < \pi/2\sqrt{K_N}$, where the sectional curvature of N is bounded above by $K_N > 0$. The gradient estimates turn out to be a powerful tool for proving existence results of harmonic maps and their heat flows on noncompact manifolds. For example, in [10], J.Y. Li used it to improve the result of P. Li and L.F. Tam [11] with a different method.

In this paper, we will first derive a gradient estimate for Dirac-harmonic maps from complete Riemannian spin manifolds into regular balls in the target manifolds, which generalizes the result for harmonic maps in [9]. As an application, we then prove a Liouville theorem for Dirac-harmonic maps under curvature conditions. We also obtain Liouville theorems under energy conditions.

When the target has nonpositive curvature, the size of the target ball is arbitrary (topological issues can be avoided by lifting to universal covers). In the presence of positive target curvature, however, we know since [12] that a restriction on the radius of the target ball is needed in order to obtain estimates. The optimal size of such a ball corresponds to an open hemisphere in the case of the standard sphere, as shown in [12]. Remarkably, we can achieve the same optimal condition on the radius $R < \pi/2\sqrt{K_N}$ as in [9] for Dirac-harmonic maps as in the original work for harmonic maps.

We can now state our gradient estimate.

Theorem 1 (*Gradient Estimate*). *Suppose the Ricci curvature of M satisfies* $\mathrm{Ric}_M \geq -\kappa$ *for some nonnegative constant* κ, *the sectional curvature* $\sec_N$ *and the curvature tensor* R^N *of N satisfy* $-b_2 \leq \sec_N \leq b_1$ *and* $\left\|\nabla R^N\right\| \leq b_3$ *respectively, where* b_i *are constants with* $b_2 \geq b_1 > 0$, $b_3 \geq 0$. *Denote*

$$b = b_2^3 + b_2^4 + b_3^2.$$

If (Φ, Ψ) *is Dirac-harmonic and* $\Phi : M^m \longrightarrow B_{y_0}(R) \subset N^n$, $R < \pi/(2\sqrt{b_1})$, *then, for any* $x_0 \in M$ *and any positive constant a, we have*

$$\sup_{B_{x_0}(a/2)} \|d\Phi\| \leq \frac{C(m,n)}{\sqrt{b_1}\cos^2(\sqrt{b_1}R)}\left(\frac{1+\sqrt{\kappa}a}{a} + \sqrt{\frac{b}{b_1}\sup_{B_{x_0}(a)}\|\Psi\|^2}\right), \tag{1.2}$$

where $C(m, n) > 0$ *is a constant depending only on the dimensions m and n.*

Remark 1. Under the hypothesis of Theorem 1, if Φ is a harmonic map and we choose $\Psi \equiv 0$, then in fact we can obtain the following global estimate for $d\Phi$:

$$\sup_M \|d\Phi\| \leq \frac{\sqrt{\min\{m,n\}\kappa}}{\sqrt{b_1}\cos(\sqrt{b_1}R)}.$$