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The comparison of two constructions of the refined analytic torsion on compact manifolds with boundary

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ABSTRACT

The refined analytic torsion on compact Riemannian manifolds with boundary has been discussed by B. Vertman (Vertman, 2009, 2008) and the authors (Huang and Lee, 2010, 2012) but these two constructions are completely different. Vertman used a double of de Rham complexes consisting of the minimal and maximal closed extensions of a flat connection and the authors used well-posed boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}, \mathcal{P}_{+,\mathcal{L}_1}$ for the odd signature operator. In this paper we compare these two constructions by using the BFK-gluing formula for zeta-determinants, the adiabatic method for stretching cylinder part near boundary and the result for comparison of eta invariants in Huang and Lee (2012) when the odd signature operator comes from a Hermitian flat connection.

1. Introduction

The refined analytic torsion was introduced by M. Braverman and T. Kappeler [1,2] on an odd dimensional closed Riemannian manifold with a flat bundle as an analytic analogue of the refined combinatorial torsion introduced by M. Farber and V. Turaev [3–6]. Even though these two objects do not coincide exactly, they are closely related. The refined analytic torsion is defined by using the graded zeta-determinant of the odd signature operator and is described as an element of the determinant line of cohomologies. Specially, when the odd signature operator comes from an acyclic Hermitian connection on a compact manifold with or without boundary, the refined analytic torsion is a complex number, whose modulus part is the Ray–Singer analytic torsion and the phase part is the ρ-invariant determined by the given odd signature operator and the odd signature operator defined by the trivial connection acting on the trivial line bundle.

The refined analytic torsion on compact Riemannian manifolds with boundary has been discussed by B. Vertman [7,8] and the authors [9,10] but these two constructions are completely different. Vertman used a double of de Rham complexes consisting of the minimal and maximal closed extensions of a flat connection. On the other hand, the authors introduced well-posed boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$ for the odd signature operator to define the refined analytic torsion on compact Riemannian manifolds with boundary. The reason is that the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ work in defining both zeta-determinants and eta invariants. Moreover, they have similar properties with the relative and absolute boundary conditions (Lemma 2.2).

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In this paper we are going to compare these two constructions when the odd signature operator comes from a Hermitian connection. The main result is Theorem 5.2 in Section 5. For comparison of the Ray–Singer analytic torsion part we are going to use the BFK-gluing formula for zeta-determinants proven in [11] and the adiabatic method for stretching cylinder part near boundary. For comparison of the eta invariant part we are going to use the result for comparison of eta invariants in [10], where the authors discussed the gluing formula of the refined analytic torsion for an acyclic Hermitian connection with respect to the well-posed boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$. Hence this work is a continuation of [10].

We now begin with the description of the odd signature operator near boundary and the refined analytic torsion with respect to the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$.

2. The refined analytic torsion on manifolds with boundary

In this section we first describe the odd signature operator $\mathcal{B}$ near boundary and introduce the well-posed boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$ for the odd signature operator. We then review the construction of the refined analytic torsions with respect to $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$ discussed in [9].

Let (M, g^M) be a compact oriented odd dimensional Riemannian manifold with boundary Y, where g^M is assumed to be a product metric near the boundary Y. We denote the dimension of M by $m = 2r - 1$. Suppose that $\rho : \pi_1(M) \to U(n)$ is a unitary representation of the fundamental group and $E = \widetilde{M} \times_\rho \mathbb{C}^n$ is the associated flat bundle, where $\widetilde{M}$ is a universal covering space of M. We choose a flat connection ∇ and extend it to a covariant differential

$$\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E).$$

Using the Hodge star operator $*_M$, we define an involution $\Gamma = \Gamma(g^M) : \Omega^\bullet(M, E) \to \Omega^{m-\bullet}(M, E)$ by

$$\Gamma\omega := i^r(-1)^{\frac{q(q+1)}{2}} *_M \omega, \quad \omega \in \Omega^q(M, E), \tag{2.1}$$

where r is given as above by $r = \frac{m+1}{2}$. It is straightforward to see that $\Gamma^2 = \mathrm{Id}$. We define the odd signature operator $\mathcal{B}$ by

$$\mathcal{B} = \mathcal{B}(\nabla, g^M) := \Gamma\nabla + \nabla\Gamma : \Omega^\bullet(M, E) \longrightarrow \Omega^\bullet(M, E). \tag{2.2}$$

Then $\mathcal{B}$ is an elliptic differential operator of order 1. Let N be a collar neighborhood of Y which is isometric to $[0, 1) \times Y$. On N a differential form ω is expressed by $\omega = \omega_{\tan} + dx \wedge \omega_{\mathrm{nor}}$, where $\omega_{\tan}$ and ω_{nor} are called the tangential and normal parts of ω, respectively. We then have a natural isomorphism

$$\Psi : \Omega^p(N, E|_N) \to C^\infty([0, 1), \Omega^p(Y, E|_Y) \oplus \Omega^{p-1}(Y, E|_Y)) \tag{2.3}$$
$$\omega_{\tan} + dx \wedge \omega_{\mathrm{nor}} \mapsto \begin{pmatrix} \omega_{\tan} \\ \omega_{\mathrm{nor}} \end{pmatrix}.$$

Using the product structure we can induce a flat connection $\nabla^Y : \Omega^\bullet(Y, E|_Y) \to \Omega^\bullet(Y, E|_Y)$ from ∇ and the Hodge star operator $*_Y : \Omega^\bullet(Y, E|_Y) \to \Omega^{m-1-\bullet}(Y, E|_Y)$ from $*_M$. We define two maps β, Γ^Y by

$$\begin{aligned} &\beta : \Omega^p(Y, E|_Y) \to \Omega^p(Y, E|_Y), \qquad \beta(\omega) = (-1)^p\omega, \\ &\Gamma^Y : \Omega^p(Y, E|_Y) \to \Omega^{m-1-p}(Y, E|_Y), \qquad \Gamma^Y(\omega) = i^{r-1}(-1)^{\frac{p(p+1)}{2}} *_Y \omega. \end{aligned} \tag{2.4}$$

It is straightforward that

$$\beta^2 = \mathrm{Id}, \qquad (\Gamma^Y)^2 = \mathrm{Id}\,. \tag{2.5}$$

Then simple computation shows that

$$\Gamma = i\beta\Gamma^Y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \nabla = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \nabla_{\partial_x} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \nabla^Y, \tag{2.6}$$

where ∂_x is the inward normal derivative to the boundary Y on N. Hence the odd signature operator $\mathcal{B}$ is expressed, under the isomorphism (2.3), by

$$\mathcal{B} = -i\beta\Gamma^Y \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla_{\partial_x} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\nabla^Y + \Gamma^Y\nabla^Y\Gamma^Y) \right\}. \tag{2.7}$$

We denote

$$\gamma := -i\beta\Gamma^Y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{A} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\nabla^Y + \Gamma^Y\nabla^Y\Gamma^Y) \tag{2.8}$$

so that $\mathcal{B}$ has the form of

$$\mathcal{B} = \gamma\,(\partial_x + \mathcal{A}) \quad \text{with } \gamma^2 = -\mathrm{Id}, \qquad \gamma\mathcal{A} = -\mathcal{A}\gamma. \tag{2.9}$$