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Vortices and the Abel–Jacobi map

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ABSTRACT

The abelian Higgs model on a compact Riemann surface Σ supports vortex solutions for any positive vortex number $d \in \mathbb{Z}$. Moreover, the vortex moduli space for fixed d has long been known to be the symmetrized d-th power of Σ, in symbols, $\mathrm{Sym}^d(\Sigma)$. This moduli space is Kähler with respect to the physically motivated metric whose geodesics describe slow vortex motion.

In this paper we appeal to classical properties of $\mathrm{Sym}^d(\Sigma)$ to obtain new results for the moduli space metric. Our main tool is the Abel–Jacobi map, which maps $\mathrm{Sym}^d(\Sigma)$ into the Jacobian of Σ. Fibres of the Abel–Jacobi map are complex projective spaces, and the first theorem we prove states that near the Bradlow limit the moduli space metric restricted to these fibres is a multiple of the Fubini–Study metric. Additional significance is given to the fibres of the Abel–Jacobi map by our second result: we show that if Σ is a hyperelliptic surface, there exist two special fibres which are geodesic submanifolds of the moduli space. Even more is true: the Abel–Jacobi map has a number of fibres which contain complex projective subspaces that are geodesic.

1. Introduction and results

Vortices in the abelian Higgs model on a Riemann surface Σ have been studied for a long time [1–8], and the vortex moduli space has also received much attention. In the so-called geodesic approximation [9,10] the motion of vortices is described by geodesics on the moduli space, with respect to a physically motivated metric. The moduli space is Kähler with respect to this metric [5,7], and a semi-explicit, local formula for the moduli space metric was also obtained in [5]. Moreover, one can give an explicit description of the moduli space as a complex manifold: this is due to the result in [6] that vortex configurations on Σ, with vortex number d, are in 1–1 correspondence with positive divisors of degree d. Hence the moduli space is the symmetrized power $\mathrm{Sym}^d(\Sigma)$, and, rather intuitively, a positive divisor $D \in \mathrm{Sym}^d(\Sigma)$ describes a configuration of vortices centred at the points in the support of D.

Here we are solely interested in the case where Σ is a compact Riemann surface. Then, viewing the moduli space as $\mathrm{Sym}^d(\Sigma)$, one can use classical methods from the theory on compact Riemann surfaces to gain further insight into the structure of the moduli space. In particular, there is a version of the Abel–Jacobi map,

$$\mathrm{AJ}\colon \mathrm{Sym}^d(\Sigma) \to \mathrm{Jac}(\Sigma), \tag{1}$$

which maps the vortex moduli space into the Jacobian $\mathrm{Jac}(\Sigma)$ of the Riemann surface Σ. The image of AJ is generally a subvariety of $\mathrm{Jac}(\Sigma)$, and the fibres are complex projective spaces. The Abel–Jacobi map is particularly useful when studying

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vortices near the so-called Bradlow limit [6]. The Bradlow limit provides an upper bound on the number of vortices that can fit on a surface Σ of finite area. In the Bradlow limit vortices are fully dissolved, i.e. the Higgs field vanishes identically. Near the Bradlow limit the Higgs field magnitude is small and therefore vortices are very spread out objects. The smallness of the Higgs field magnitude suggests that one should expand interesting quantities, such as the moduli space metric, in terms of the Higgs field magnitude.

The usefulness of the Abel–Jacobi map near the Bradlow limit is a consequence of the fact that in the strict Bradlow limit the vortex moduli space coincides with $\mathrm{Jac}(\Sigma)$. The moduli space metric in the Bradlow limit was recently shown in [11] to be the flat metric on the torus $\mathrm{Jac}(\Sigma)$. Near the Bradlow limit the fibres of AJ are metrically small, and the lowest order contribution to the moduli space metric is due to vortex motion in the image of AJ. This contribution was also worked out in [11], and it was found to be degenerate at those $\mathfrak{p} \in \mathrm{Jac}(\Sigma)$ for which $\mathrm{AJ}^{-1}\{\mathfrak{p}\}$ has positive dimension. This leads to the question what contribution to the moduli space metric is due to vortex motion in the fibres of AJ. The answer to this question is our first theorem.

Theorem 1. *Let $\mathfrak{p} \in \mathrm{Jac}(\Sigma)$ be in the image of AJ, and let $k \in \mathbb{N}$ be such that $\mathrm{AJ}^{-1}\{\mathfrak{p}\} \cong \mathbb{CP}^k$. Near the Bradlow limit the leading order contribution to the moduli space metric, restricted to $\mathrm{AJ}^{-1}\{\mathfrak{p}\}$, is a multiple of the Fubini–Study metric on $\mathbb{CP}^k$.*

This result generalizes the work of [12], where vortices on $\mathbb{CP}^1$ were studied near the Bradlow limit, and the moduli space metric was found to be a multiple of the Fubini–Study metric. Note that $\mathrm{Sym}^d(\mathbb{CP}^1) \cong \mathbb{CP}^d$, and since the Jacobian of $\mathbb{CP}^1$ is a point, the Abel–Jacobi map is trivial. Hence, the result of [12] is a special case of Theorem 1.

While Theorem 1 serves to alleviate the degeneracy of the moduli space metric found in [11], the splitting of vortex motion into directions in the image and fibre of AJ does not appear to be very natural from a physical point of view. However, if a fibre of AJ is a geodesic submanifold of the moduli space, then it is sensible to study the dynamics of vortex configurations that correspond to points in this fibre. Our second result is that on a hyperelliptic Riemann surface there exist two fibres of AJ that are indeed geodesic.

Before we can express the previous statement as a theorem, we need to introduce some notation: by K_Σ we denote the canonical line bundle of the Riemann surface Σ. A hyperelliptic Riemann surface Σ is defined by the existence of a holomorphic projection map

$$\pi : \Sigma \to \mathbb{CP}^1, \tag{2}$$

which is 2–1. If we regard a point $p \in \mathbb{CP}^1$ as a divisor of degree one, then the pulled-back divisor $\pi^{-1}(p)$ consists of the two points, counted with multiplicity, in the preimage of π. The holomorphic line bundle $\mathcal{O}(\pi^{-1}(p))$ on Σ is independent of p, up to isomorphism. The existence of the map π also leads to a natural holomorphic automorphism of Σ, defined by exchanging the two points in $\pi^{-1}(p)$. This automorphism is referred to as the hyperelliptic involution of Σ. Now we are ready to state our next theorem.

Theorem 2. *Let Σ be a hyperelliptic Riemann surface, equipped with a metric such that the hyperelliptic involution of Σ is an isometry. Then $\mathbb{P}\mathrm{H}^0(\Sigma, K_\Sigma)$ and $\mathbb{P}\mathrm{H}^0(\Sigma, \mathcal{O}(\pi^{-1}(p)))$, $p \in \mathbb{CP}^1$, are geodesic submanifolds of the moduli space.*

As usual, $\mathrm{H}^0(\Sigma, K_\Sigma)$ denotes the space of holomorphic sections of K_Σ, and $\mathbb{P}\mathrm{H}^0(\Sigma, K_\Sigma)$ is its projectivization, and analogously for $\mathcal{O}(\pi^{-1}(p))$. It is a standard observation that $\mathbb{P}\mathrm{H}^0(\Sigma, K_\Sigma)$ and $\mathbb{P}\mathrm{H}^0(\Sigma, \mathcal{O}(\pi^{-1}(p)))$ can be identified with fibres of the Abel–Jacobi map, and we will review this in Section 2. We stress that Theorem 2 holds in general, not only near the Bradlow limit. The fact that the fibre $\mathbb{P}\mathrm{H}^0(\Sigma, \mathcal{O}(\pi^{-1}(p)))$ is a geodesic submanifold of the vortex moduli space is also a consequence of Proposition 7.1 in [13], for which an isometry on the moduli space is required.

Theorem 2 is a special case of Lemma 2, which we will establish in Section 4. In this introduction we omit a precise statement of Lemma 2 since this requires some additional technical preparations. In summary, Lemma 2 identifies a number of complex projective spaces that embed into the vortex moduli space as geodesic submanifolds. These complex projective spaces are linear systems of divisors on Σ. However, these linear systems are not necessarily complete, i.e. they may not fill out entire fibres of the Abel–Jacobi map.

The structure of this paper is as follows. In Section 2 we review classification results for holomorphic line bundles on compact Riemann surfaces. This review includes the definition of the Abel–Jacobi map and general properties of its fibres. In Section 3 we study the Bogomolny equations near the Bradlow limit. We introduce a suitable description of the moduli space near the Bradlow limit, and based on this we prove Theorem 1. In Section 4 we review standard properties of hyperelliptic Riemann surfaces and we use them to prove Lemma 2, and consequently Theorem 2.

2. Holomorphic line bundles on Riemann surfaces

In this section we summarize the standard classification results for holomorphic line bundles in terms of the Picard and Jacobian varieties. We also review how holomorphic line bundles can be characterized equivalently by divisors and by Dolbeault operators. For detailed derivations and proofs we refer to standard textbooks such as [14–17]. The main purpose of this section is to introduce notation, and the reader familiar with the theory of line bundles on compact Riemann surfaces may well want to skip to Section 3.