Journal of Geometry and Physics 76 (2014) 256–262

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Capillary surfaces in a cone

Rafael López [a,*], Juncheol Pyo [b]

[a] Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain
[b] Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea

ARTICLE INFO

Article history:
Received 3 October 2012
Received in revised form 7 September 2013
Accepted 23 October 2013
Available online 8 November 2013

MSC:
53A10
49Q10
76B45
76D45

Keywords:
Capillary surface
Spherical reflection
Mean curvature

ABSTRACT

We show that a capillary surface in a solid cone, that is, a surface that has constant mean curvature and for which the surface boundary meets the boundary of the cone at a constant angle, is radially graphical if the mean curvature is non-positive with respect to the Gauss map pointing towards the domain bounded by the surface and the boundary of the cone. In the particular case in which the cone is circular, we prove that the surface is a spherical cap or a planar disc. The proofs are based on an extension of the Alexandrov reflection method using inversions about spheres.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

Consider a given amount of liquid deposited on a solid substrate with a conical shape, as shown in Fig. 1. In absence of gravity, we analyze the equilibrium configurations when the drop reaches a critical interfacial area. Let (x, y, z) be the usual coordinates of $\mathbb{R}^3$, where z indicates the vertical direction. Let $\mathbb{S}^2$ be the unit sphere centered at the origin O of $\mathbb{R}^3$ and let $D \subset \mathbb{S}^2$ be a simply connected domain of a hemisphere of $\mathbb{S}^2$. We denote by C_D the cone defined by the union of all half-lines starting at O passing through all points of D and C_Γ the boundary of C_D, that is, the union of the rays that start at O through all points of $\Gamma = \partial D$. The point O is called the vertex of the cone. The cone C_D is called a circle of opening angle $2\varphi \in (0, \pi)$ if $\Gamma \subset \mathbb{S}^2$ is a circle of radius $\sin(\varphi)$. A liquid drop in the cone is viewed as the closure of a domain $\Omega \subset \mathbb{R}^3$ confined in the cone C_D and whose boundary $\partial\Omega$ intersects C_Γ. The boundary $\partial\Omega$ of Ω is written as $\partial\Omega = T \cup S$, where $T \subset C_\Gamma$, $S = \partial\Omega \setminus T$, and $\partial\Omega$ is not smooth along its boundary ∂S. Physically, T is the region wetted by the liquid drop $\overline{\Omega}$ in C_Γ.

In the absence of gravity, according to the Young–Laplace equation, the shape of a liquid drop in equilibrium is characterized by the following properties: (1) the surface S has a constant mean curvature; and (2) the angle γ at which S meets C_Γ is constant. Here, $\cos\gamma = \langle N, N_C \rangle$ along ∂S, where N and N_C denote the unit normal vector fields of S and C_Γ that point towards Ω and C_D, respectively. Using the above notation, we provide the following definition.

Definition 1. A capillary surface on a cone C_D is a compact embedded surface that has a constant mean curvature, such that $int(S) \subset C_D$ and $\partial S \subset C_\Gamma$, and that meets the cone C_Γ at a constant angle.

* Corresponding author. Tel.: +34 958243396.
E-mail addresses: rcamino@ugr.es (R. López), jcpyo@pusan.ac.kr (J. Pyo).

0393-0440/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.geomphys.2013.10.021

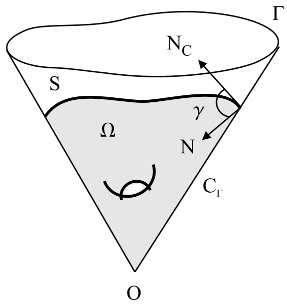

Fig. 1. A drop Ω supported in a cone C_Γ. The interface S is a capillary surface, which means that S has constant mean curvature and the angle γ between the unit normal vectors N and N_C for S and C_Γ, respectively, is constant along the boundary curve ∂S.

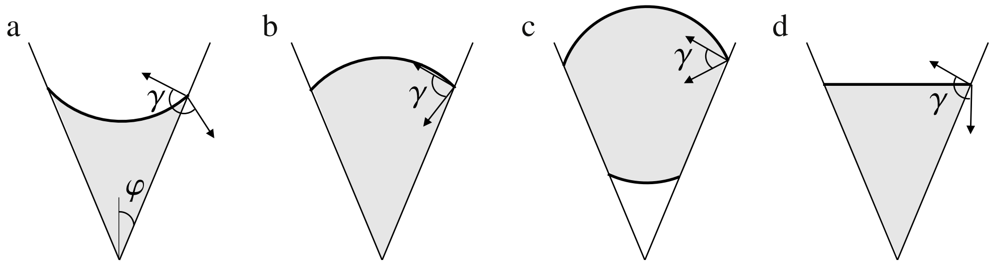

Fig. 2. Different configurations for capillary spheres and discs in a circular cone.

In principle, a capillary surface in a cone can have higher topology, as well as any number of boundary components. Physically, capillary surfaces arise as the interface of an incompressible liquid in a container [1]. Liquid drops in a cone have previously been described [2].

Examples of capillary surfaces in a circular cone C_D include pieces of spheres and planar discs (Fig. 2). Consider a round disc $D \subset \mathbb{S}^2$ centered at the north pole of radius $\sin(\varphi) \in (0, 1)$. Consider a sphere Σ centered at the positive z-axis of radius sufficiently large so that Σ intersects C_Γ. Then $S = \Sigma \cap C_D$ is a capillary surface in C_D. Depending on the values of γ and φ, we have the following cases. (a) If $\gamma > \pi/2 + \varphi$, S is a concave interface; (b) if $\pi/2 - \varphi \leq \gamma < \pi/2 + \varphi$, S is a convex interface; (c) if $\gamma < \pi/2 - \varphi$, S is bounded by two spherical caps; and (d) if $\gamma = \pi/2 + \varphi$, we have a flat interface.

For a general cone, a first set of examples of capillary surfaces includes non-parametric surfaces, that is, surfaces with a one-to-one central projection on D. We say that the surface S is a radial graph, which implies that any half-line starting at O intersects with S at one point at most. Results for the existence and uniqueness of radial graphs with constant mean curvature have been published elsewhere [3–7]. Radial graphs and capillary surfaces in a cone are examples analogous to (vertical) graphs on a plane and capillary surfaces in a vertical cylinder if we move the vertex O of the cone to infinity. For a convex cone C_Γ, Choe and Park have shown that if a parametric capillary surface S meets C_Γ orthogonally, then S is part of a sphere [8]. Ros and Vergasta proved that S has some topological restrictions if S is stable [9]. Vogel considered a drop Ω in a cone C_D such that its adherence $\overline{\Omega}$ contains the vertex O, and proved that the surface S is a radial graph if the mean curvature H is non-positive with respect to the unit normal vector field pointing to Ω [10].

This is our starting point. We investigate the shape of a capillary surface in a cone and how the boundary of the surface imposes restrictions on its shape. We pose the following questions:

1. Can a capillary surface exist in a cone C_D whose boundary is null-homologous in $C_\Gamma \setminus \{O\}$? We do not know of explicit examples and we hope that the answer is no, at least if the surface is stable.
2. If the boundary ∂S is homologous to Γ in $C_\Gamma \setminus \{O\}$, is S a radial graph?
3. Under what conditions is a capillary surface part of a sphere or a plane?
4. Under what conditions is a capillary surface a bridge, whereby the surface has the topology of a portion of a cylinder bounded by two simple closed curves?

We extend the results of Vogel [10] assuming only non-positivity for the mean curvature H. Recall that a compact embedded surface S contained in a cone C_D and $\partial S \subset C_\Gamma$ defines a closed 3-dimensional domain $\Omega \subset C_D$ such that its boundary $\partial\Omega$ is $\partial\Omega = T \cup S$ with $T \subset C_\Gamma$.

Using the above notation, our main result is the following theorem.

Theorem 2. *Let S be a connected capillary surface supported in a cone C_D. We fix the unit normal vector field N of S pointing towards the bounded domain Ω. If $H \leq 0$, then S is a radial graph and the boundary ∂S has only one connected component that is homologous to Γ in $C_\Gamma \setminus \{O\}$. In the particular case in which the cone is circular, S is a planar disc or a spherical cap.*

Download English Version:

https://daneshyari.com/en/article/1893387

Download Persian Version:

https://daneshyari.com/article/1893387

Daneshyari.com