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Coherent states attached to the spectrum of the Bochner Laplacian for the Hopf fibration

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ABSTRACT

We construct coherent states attached to spectrum of the Bochner Laplacian for the Hopf fibration. This enables us to obtain a closed form for the expression of the coherent states attached to Landau levels on the Riemann sphere. We also obtain an identity for certain orthogonal polynomials.

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1. Introduction

Coherent states have attracted much attention in the recent decades. They are a useful mathematical framework for dealing with the connection between classical and quantum formalisms. In general, coherent states are a specific overcomplete family of vectors in a certain Hilbert space. For various definitions, see the review article [1] and the references therein.

In a previous work [2] we have been concerned with a charged particle evolving in the Riemann sphere under the influence of a normal uniform magnetic field. We have attached to eigenvalues of the corresponding Hamiltonian a family of coherent states by following a generalization of the conventional boson coherent states when written as sums of eigenstates of the number operator.

In this paper, we deal with an analogous question in the context of the Horizontal Laplacian for the Hopf fibration when acting on smooth sections of certain complex line bundles associated to the $U(1)$-principal bundle over the two-sphere. Indeed, we construct for each eigenvalue of this Laplacian a family of coherent states labelled by elements of the Lie group $SU(2)$. The expressions of the constructed coherent states provide us, when considering the quotient of $SU(2)$ by $U(1)$, with a closed form for the expressions of the coherent states attached to Landau levels on the Riemann sphere. This also leads us to an identity for certain orthogonal polynomials.

We organize the paper as follows. In Section 2, we describe the formalism of constructing coherent states. In Section 3, we review the Hopf fibration and the corresponding horizontal Laplacian (Bochner Laplacian). In Section 4, some required facts on the spectral analysis of this Laplacian are discussed. Section 5 deals with the construction of the coherent states attached to the eigenvalues of the Bochner Laplacian. In Section 6 we discuss a closed form for the expression of coherent states attached to Landau levels on the Riemann sphere.

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2. Coherent states

Let (X, μ) be a measure space and let $\mathcal{A}^2 \subset L^2(X,\mu)$ be a closed subspace of finite dimension d. Let $\{\Phi_n\}_{n=1}^{n=d}$ be an orthogonal basis of $\mathcal{A}^2$ satisfying, for arbitrary $x \in X$,

$$\omega_d(x) := \sum_{n=1}^{d} \frac{|\Phi_n(x)|^2}{\rho_n} < +\infty,$$

where $\rho_n := \|\Phi_n\|^2_{L^2(X)}$. Define

$$K(x,y) := \sum_{n=1}^{d} \frac{\Phi_n(x)\overline{\Phi_n(y)}}{\rho_n}, \quad x, y \in X.$$

Then, $K(x,y)$ is a reproducing kernel, $\mathcal{A}^2$ is the corresponding reproducing kernel Hilbert space and $\omega_d(x) = K(x,x)$, $x \in X$.

Let $\mathcal{H}$ be another Hilbert space with $\dim \mathcal{H} = d$ and $\{\phi_n\}_{n=1}^{n=d}$ be an orthonormal basis of $\mathcal{H}$. Therefore, for $x \in X$, define

$$|x\rangle := (\omega_d(x))^{-\frac{1}{2}} \sum_{n=1}^{d} \frac{\Phi_n(x)}{\sqrt{\rho_n}} \phi_n. \tag{2.1}$$

The choice of the Hilbert space $\mathcal{H}$ defines a quantization of the set $X = \{x\}$ by the coherent states $|x\rangle$, via the inclusion map: $x \to |x\rangle$ from X into $\mathcal{H}$.

By definition, it is straightforward to show that $\langle x|x\rangle = 1$ and the coherent state transform $\mathcal{W} : \mathcal{H} \to \mathcal{A}^2 \subset L^2(X,\mu)$ defined by

$$\mathcal{W}[\phi](x) := (\omega_d(x))^{\frac{1}{2}} \langle x|\phi\rangle \tag{2.2}$$

is an isometry. Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$\langle \phi|\psi\rangle_{\mathcal{H}} = \langle \mathcal{W}[\phi]\,|\,\mathcal{W}[\psi]\rangle_{L^2(X)} = \int_X d\mu(x)\,\omega_d(x)\,\langle\phi|x\rangle\langle x|\psi\rangle$$

and thereby we have a resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_X d\mu(x)\,\omega_d(x)\,|x\rangle\langle x|, \tag{2.3}$$

where $\omega_d(x)$ appears as a weight function.

3. The horizontal Laplacian for the Hopf fibration

Let us denote $G = SU(2)$ and $H = U(1)$

$$H = \left\{ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \varphi \in \mathbb{R} \right\}.$$

A general element of G is

$$g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad \alpha\overline{\alpha} + \beta\overline{\beta} = 1.$$

Writing α and β in terms of their real and imaginary parts as $\alpha = x_1 + ix_2$ and $\beta = x_3 + ix_4$. Then $SU(2)$ can be identified with the unit sphere in $\mathbb{R}^4$.

$$S^3 = \left\{ (x_1, x_2, x_3, x_4) ; (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1 \right\}.$$

We can define a map $\pi : G \to S^2$ by $\pi(g) = g\sigma_3 g^{-1}$, where σ_3 is the matrix diag $(1, -1)$. The kernel of the map π is precisely $U(1)$; thus we have a $U(1)$ fibration over $S^2 = SU(2)/U(1)$ in S^3:

$$S^1 \to S^3 \to S^2$$

which is known as the Hopf fibration.

Now, following [3] let

$$D_{i,j} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \quad 1 \le i, j \le 4$$

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