Journal of Geometry and Physics 75 (2014) 55–70

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Deformation quantization with separation of variables of an endomorphism bundle

Alexander Karabegov *

Department of Mathematics, Abilene Christian University, ACU Box 28012, Abilene, TX 79699-8012, United States

ARTICLE INFO

Article history:
Received 27 July 2013
Accepted 31 August 2013
Available online 10 September 2013

MSC:
53D55
81T18

Keywords:
Kähler–Poisson manifold
Endomorphism bundle
Deformation quantization with separation of variables
Feynman graph

ABSTRACT

Given a holomorphic Hermitian vector bundle E and a star-product with separation of variables on a pseudo-Kähler manifold, we construct a star product on the sections of the endomorphism bundle of the dual bundle E^* which also has the appropriately generalized property of separation of variables. For this star product we prove a generalization of Gammelgaard's graph-theoretic formula.

1. Introduction

Deformation quantization on a Poisson manifold $(M, \{\cdot,\cdot\})$ is an associative product on the space $C^\infty(M)[[\nu]]$ of ν-formal smooth complex-valued functions given by the formula

$$f * g = \sum_{r\geq 0} \nu^r C_r(f,g), \tag{1}$$

where C_r are bidifferential operators on M, $C_0(f,g) = fg$, and

$$C_1(f,g) - C_1(g,f) = i\{f,g\}.$$

The product $*$ is called a star product. It is assumed that star products are normalized, i.e., the unit constant function $\mathbf{1}$ is the unity of a star product, $f * \mathbf{1} = \mathbf{1} * f = f$. Two star products $*_1, *_2$ on $(M, \{\cdot,\cdot\})$ are called equivalent if there exists a formal differential operator $T = 1 + \nu T_1 + \cdots$ on M such that $T(f *_1 g) = Tf *_2 Tg$. Star products on M can be restricted (localized) to any open subset of M. A star product on M can be extended to $C^\infty(M)[\nu^{-1}, \nu]]$, the space of formal Laurent series of functions with a finite polar part, $f = \nu^s f_s + \nu^{s+1} f_{s+1} + \cdots$, where s is a possibly negative integer.

In the theory of deformation quantization there are general existence and classification results, specific constructions of star products, and explicit formulas for star products. The problem of existence and classification of star products up to equivalence on an arbitrary Poisson manifold was stated in [1] and settled in [2] by Kontsevich. In [3] Fedosov gave a geometric construction of star products from every equivalence class on an arbitrary symplectic manifold. There are

* Tel.: +1 325 674 2175; fax: +1 325 674 6753.
E-mail address: axk02d@acu.edu.

0393-0440/$ – see front matter
http://dx.doi.org/10.1016/j.geomphys.2013.08.018

star products on Kähler manifolds with the property of separation of variables which originate in the context of Berezin's quantization, such as the Berezin and Berezin–Toeplitz star products (see [4–8]). Star products with separation of variables on Kähler manifolds are bijectively parameterized by formal Kähler forms.

When the existence of a specific star product is established, finding an explicit formula for that product may still be a challenging problem. For almost two decades since [1], explicit formulas had been known only for a few examples of invariant star products on homogeneous symplectic spaces such as Weyl, *pq*-, *qp*-, Wick, and anti-Wick star-products on linear symplectic spaces, and star-products on complex projective spaces and Grassmann manifolds (see [9–11]). For non-invariant star products there are explicit formulas expressed in terms of directed graphs. The first such formula is the celebrated Kontsevich's formula for a star product on $\mathbb{R}^n$ equipped with an arbitrary Poisson structure (see [2]). There are several explicit graph-theoretic formulas for star products with separation of variables on Kähler manifolds. In [12] Reshetikhin and Takhtajan gave a graph-theoretic formula for a star product on an arbitrary Kähler manifold which was based upon a formal interpretation of integral formulas for Berezin's quantization. However, the star product given by their explicit formula is not normalized. Inspired by their work, Gammelgaard gave in [13] an explicit formula for the star product with separation of variables with an arbitrary parameterizing formal Kähler form. Gammelgaard's formula specifies directly to the Berezin–Toeplitz star product owing to the explicit description of its parameterizing form from [14]. Recently Hao Xu found in [15] a graph-theoretic formula for Berezin's star product and calculated its parameterizing form in [16].

It is worth mentioning that so far there are no known explicit formulas for Fedosov's star products.

Deformation quantization of endomorphism bundles was used in the proofs of the index theorem for deformation quantization and its generalizations (see [17–20]). Matrix-valued symbols and related quantizations were considered in [21–24].

Star products with separation of variables on the sections of the endomorphism bundle of a holomorphic Hermitian vector bundle on a Kähler manifold were constructed in [25] using Fedosov's approach. In this paper we give an alternative construction of such star products in the spirit of [7] and prove a generalized Gammelgaard's graph-theoretic formula for these star products. To this end we generalize the proof of Gammelgaard's formula from [26]. Our sign conventions differ from those in [13,26].

2. Deformation quantizations with separation of variables

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold endowed with a complex structure such that the Poisson tensor on M is of type $(1, 1)$ with respect to the complex structure. We call M a Kähler–Poisson manifold. In the rest of the paper we denote by m the complex dimension of M. In local holomorphic coordinates $z^k, \bar{z}^l$ we write the Kähler–Poisson tensor as g^{lk}, so that

$$\{f, g\} = ig^{lk} \left(\frac{\partial f}{\partial z^k} \frac{\partial g}{\partial \bar{z}^l} - \frac{\partial g}{\partial z^k} \frac{\partial f}{\partial \bar{z}^l} \right). \tag{2}$$

If the tensor g^{lk} is nondegenerate, its inverse g_{kl} is a pseudo-Kähler metric tensor on M.

A star product (1) on a Kähler–Poisson manifold M is called a star product with separation of variables if the bidifferential operators C_r differentiate their first argument only in antiholomorphic directions and the second argument only in holomorphic ones. Equivalently, if f and g are local functions on M and f is holomorphic or g is antiholomorphic, then

$$f * g = fg. \tag{3}$$

Star products with separation of variables originate in the context of Berezin's quantization (see [4]). It was shown in [7,8] that star products with separation of variables exist on arbitrary pseudo-Kähler manifolds. In the terminology of [8], star products with separation of variables are of anti-Wick type.

It is not yet known whether star products with separation of variables exist on arbitrary Kähler–Poisson manifolds. Examples of such star products on Kähler–Poisson manifolds with degenerate Kähler–Poisson tensors are given in [5,27,28].

Deformation quantizations with separation of variables on a pseudo-Kähler manifold M were classified in [7]. Denote by ω_{-1} the pseudo-Kähler form on M. Consider a formal form

$$\omega = \frac{1}{\nu}\omega_{-1} + \omega_0 + \nu\omega_1 + \cdots,$$

where $\omega_r,\ r \geq 0$, are possibly degenerate closed forms of type $(1, 1)$ with respect to the complex structure. The star products with separation of variables on M are bijectively parameterized by such formal forms. The star product with separation of variables $*$ parameterized by a formal form ω is completely characterized by the property that

$$\frac{\partial \Phi}{\partial z^k} * f = \frac{\partial \Phi}{\partial z^k} f + \frac{\partial f}{\partial z^k}, \quad 1 \leq k \leq m, \tag{4}$$

for any local potential

$$\Phi = \frac{1}{\nu}\Phi_{-1} + \Phi_0 + \cdots$$

of the form ω, so that $\omega = i\partial\bar{\partial}\Phi$. Equivalently, $*$ is completely characterized by the property that locally

$$f * \frac{\partial \Phi}{\partial \bar{z}^l} = \frac{\partial \Phi}{\partial \bar{z}^l} f + \frac{\partial f}{\partial \bar{z}^l}. \tag{5}$$