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Gauge networks in noncommutative geometry

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ABSTRACT

We introduce gauge networks as generalizations of spin networks and lattice gauge fields to almost-commutative manifolds. The configuration space of quiver representations (modulo equivalence) in the category of finite spectral triples is studied; gauge networks appear as an orthonormal basis in a corresponding Hilbert space. We give many examples of gauge networks, also beyond the well-known spin network examples. We find a Hamiltonian operator on this Hilbert space, inducing a time evolution on the C^*-algebra of gauge network correspondences.

Given a representation in the category of spectral triples of a quiver embedded in a spin manifold, we define a discretized Dirac operator on the quiver. We compute the spectral action of this Dirac operator on a four-dimensional lattice, and find that it reduces to the Wilson action for lattice gauge theories and a Higgs field lattice system. As such, in the continuum limit it reduces to the Yang–Mills–Higgs system. For the three-dimensional case, we relate the spectral action functional to the Kogut–Susskind Hamiltonian.

1. Introduction

We develop a formalism of *gauge networks* that bridges between three apparently different notions: the theory of spin networks in quantum gravity, lattice gauge theory, and the almost-commutative geometries used in the construction of particle physics models via noncommutative geometry.

The main idea behind the spin network approach to quantum gravity is that a space continuum is replaced by quanta of space carried by the vertices of a graph and quanta of areas, representing the boundary surface between two adjacent quanta of volume, carried by the graph edges. The metric data are encoded by holonomies described by $SU(2)$ representations associated to the edges with intertwiners at the vertices, [1,2].

On the other hand, in the noncommutative geometry approach to models of matter coupled to gravity, one considers a noncommutative geometry that is locally a product of an ordinary 4-dimensional spacetime manifold and a *finite spectral triple*. A spectral triple, in general, is a noncommutative generalization of a compact spin manifold, defined by the data (A, H, D) of an involutive algebra A with a representation as bounded operators on a Hilbert space H, and a Dirac operator, which is a densely defined self-adjoint operator with compact resolvent, satisfying the compatibility condition that commutators with elements in the algebra are bounded. In the finite case, both A and H are finite dimensional: such a space corresponds to a metrically zero dimensional noncommutative space. A product space of a finite spectral triple and an ordinary manifold (also seen as a spectral triple) is known as an *almost-commutative geometry*. There is a natural action

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functional, the *spectral action*, on such spaces, whose asymptotic expansion recovers the classical action for gravity coupled to matter, where the matter sector Lagrangian is determined by the choice of the finite noncommutative space, [3–6].

Just as the notion of a spin network encodes the idea of a discretization of a 3-manifold, one can consider a similar approach in the case of the almost-commutative geometries and "discretize" the manifold part of the geometry, transforming it into the data of a graph, with finite spectral triples attached to the vertices and morphisms attached to the edges. This is the basis for our definition of gauge networks, which can be thought of as *quanta of noncommutative space*. While we mostly restrict our attention to the gauge case, where the Dirac operators in the finite spectral triples are trivial, the same construction works more generally. We show that the manifold Dirac operator of the almost-commutative geometry can be replaced by a discretized version defined in terms of the graph and of holonomies along the edges.

In lattice gauge theory, the Wilson action defined in terms of holonomies recovers, in the continuum limit, the Yang–Mills action, [7]. We show that the spectral action of the Dirac operator on a gauge network recovers the Wilson action with additional terms that give the correct action for a lattice gauge theory with a Higgs field in the adjoint representation, [8,9].

In Section 2 we construct a category whose objects are *finite spectral triples* and whose morphisms are pairs of an algebra morphism and a unitary operator with a compatibility condition, and a subcategory made of those finite spectral triples that have trivial Dirac operator. We give some explicit examples, including those related to Yang–Mills theory and to the Standard Model. Using the Artin–Wedderburn theorem, one can write the algebras as sums of matrix algebras and describe the morphisms in terms of Bratteli diagrams and of more general braid Bratteli diagrams, which keep into account the permutations of blocks of the same dimension. We then introduce the main objects of our constructions, which are representations of quivers (oriented graphs) in the category of finite spectral triples described above. The configuration space $\mathcal{X}$ is the space of such representations and we also consider its quotient by a natural group $\mathcal{G}$ of symmetries given by the invertible morphisms at each vertex of the graph. This quotient can be understood as taking equivalence classes of quiver representations in the category of finite spectral triples. The space $\mathcal{X}$ and the $\mathcal{G}$-invariants of $L^2(\mathcal{X})$ are described more explicitly using the orbit-stabilizer theorem, the Peter–Weyl theorem for compact Lie groups, and its extension to homogeneous spaces. An orthonormal basis is given in terms of the intertwiners at vertices. Thus, the data of a *gauge network* can be defined in terms of a quiver representation in the category of finite spectral triples with vanishing Dirac operator, carrying unitary Lie group representations along the edges and intertwiners at vertices. We show that the data obtained in this way, in the case where the pair (A_v, H_v) at each vertex is $(M_N(\mathbb{C}), \mathbb{C}^N)$ with trivial Dirac operator, recovers the case of $U(N)$ spin networks. Other examples of gauge networks are discussed in this section, including abelian spin networks, $U(N)$ spin networks, and some non-spin-network examples with trivial Hilbert space (the representation in the spectral triple datum is not assumed to be faithful), where the Peter–Weyl decomposition of $L^2(\mathcal{X})$ can be described in terms of Gelfand–Tsetlin diagrams.

In Section 3 we give a categorical formulation by introducing morphisms between gauge networks in the form of correspondences defined by bimodules. We also define a C^*-algebra of gauge network correspondences, and a time evolution, where the Hamiltonian is an operator on $L^2(\mathcal{X})$ defined as a sum of quadratic Casimir operators of the Lie groups $\mathcal{U}(A_{t(e)})$. This makes the noncommutative geometries described by gauge networks dynamical.

In Section 4 we introduce a notion of (discretized) Dirac operator for a representation (in the category of spectral triples) of a quiver embedded in a Riemannian spin manifold, and we show that in the lattice case, in the continuum limit where the lattice size goes to zero, this recovers the usual geometric Dirac operator on a manifold. We also consider Dirac operators twisted by gauge potentials. These Dirac operators turn the quiver representations into spectral triples. We then consider the spectral action, computed for a quiver that is a four-dimensional lattice. We show that it reduces to the Wilson action for lattice gauge theory and a Higgs field lattice system, with the Higgs field in the adjoint representation. In the case of a 3-dimensional lattice we recover the Kogut–Susskind Hamiltonian. We finish the section with a proposal for an extension of our formalism from gauge networks to gauge foams, which we hope to return to in future work.

2. Quiver representations and finite spectral triples

We introduce the notion of a gauge network, thereby generalizing spin networks to quanta of *noncommutative* space. We adopt a (noncommutative) differential geometrical point of view and take spectral triples as our starting point.

2.1. Finite-dimensional algebra representations and finite spectral triples

We start by introducing a category of finite-dimensional algebras, together with a representation on a Hilbert space.

Definition 1. The category $\mathcal{C}_0$ has as objects triples (A, λ, H) where A is a finite-dimensional C^*-algebra, and λ is a $*$-representation on an inner product space H. A morphism in $\mathrm{Hom}((A_1, H_1), (A_2, H_2))$ is given by a pair (ϕ, L) consisting of a unital $*$-algebra map $\phi : A_1 \to A_2$ and a unitary $L : H_1 \to H_2$ such that

$$L\lambda_1(a)L^* = \lambda_2(\phi(a)); \quad (\forall a \in A_1). \tag{1}$$

An alternative definition of the above category $\mathcal{C}_0$ is as a category of finite spectral triples (A, H, D) with vanishing Dirac operator $D = 0$.