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Path space connections and categorical geometry

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ABSTRACT

We develop a new differential geometric structure using category theoretic tools that provides a powerful framework for studying bundles over path spaces. We study a type of connection forms, given by Chen integrals, over path spaces by placing such forms within a category-theoretic framework of principal bundles and connections. A new notion of 'decorated' principal bundles is introduced, along with parallel transport for such bundles, and specific examples in the context of path spaces are developed.

1. Introduction

In this paper we develop a theory of categorical geometry and explore specific examples involving geometry over spaces of paths. Our first objective is to develop a framework that encodes special properties, such as parametrization-invariance, of connection forms on path spaces. For this paper we focus on the case of connections over path spaces given by first-order Chen integrals. We then develop a theory of 'decorated' principal bundles and parallel-transport in such bundles. These constructions all sit naturally inside a framework of categorical connections that we develop. A background motivation is to develop a framework that provides a unified setting for both ordinary gauge theory, governing interactions between point particles, and higher gauge theory, governing the interaction of string-like, or higher-dimensional, extended objects.

There is a considerable current literature (as we cite in a paragraph below) combining geometric ideas and category theoretic structures. However, our work offers several new ideas; these include:

- our formulation of the notion of a *categorical connection*, closer in spirit to the traditional notion of parallel-transport but more general than the formulations that are in use in the existing literature;
- a new notion of *decorated bundles* that provides a natural and rich class of examples of categorical bundles.

Our development of a powerful framework of categorical principal bundles is different from other works, with the action of the categorical structure group playing a more explicit direct role, analogous to the case of classical principal bundles.

Our constructions are not developed for the sake of abstract constructions, but rather as a natural framework that expresses the essential elements of a variety of examples. These examples include connections on bundles over path spaces as well as a new but natural notion of 'decorated' bundles that we introduce. The study of such bundles is motivated by gauge

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theories in which points in a bundle are replaced by paths, on the bundle, decorated by elements of a group. Our theory of categorical connections then provides a natural framework for formulating the notion of parallel-transport of such decorated paths. Decorated principal bundles provide a framework for generating hierarchies of examples of categorical bundles and connections over higher dimensional (that is, iterations of) path spaces.

We now summarize our results through an overview of the paper. All categories we work with are 'small': the objects and morphisms form sets.

- In Section 2 we study a connection form $\omega_{(A,B)}$ on a principal bundle over a path space. This connection form is invariant under reparametrization of the paths. We then describe the connection form $\omega_{(A,B)}$:

$$\omega_{(A,B)}(\tilde{v}) = A\big(\tilde{v}(t_0)\big) + \tau\left[\int_{t_0}^{t_1} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt\right], \tag{1.1}$$

  specifying the meaning of all the terms involved here. Briefly and roughly put, $\omega_{(A,B)}$ is a 1-form on path space obtained by a 'point-evaluation' of a traditional 1-form (the first term on the right) and a first-order Chen integral (the second term on the right). Next in Proposition 2.2 we prove that this connection form is also invariant under reparametrization of paths, and so induces a connection form on the space of reparametrization-equivalence classes of paths. In Proposition 2.2 we show that $\omega_{(A,B)}$ does have properties analogous to traditional connection forms on bundles.
- In Section 3 we consider equivalence classes of paths, identifying paths that differ from each other by erasure of backtracked segments (that is, a composite path $c_2\bar{a}ac_1$, where $\bar{a}$ is the reverse of a, is considered equivalent to c_2c_1). The results of Section 3, especially Theorem 3.1, show that $\omega_{(A,B)}$ specifies a connection form on the space of backtrack-erasure equivalence classes of paths. This is the result that links the geometry with category theory: it makes it possible to view $\omega_{(A,B)}$ as specifying a connection form over a category whose morphisms are backtrack-erased paths. In Theorem 3.2 we prove that $\omega_{(A,B)}$ respects another common way of identifying paths: paths γ_1 and γ_2 are said to be 'thin homotopic' if one can obtain γ_2 from γ_1 by means of a homotopy that 'wiggles' γ_1 along itself. Thus, $\omega_{(A,B)}$ specifies a connection also over the space of thin-homotopy equivalence classes of paths.
- In Section 4 we study the notion of a *categorical group*; briefly, this is a category whose object set and morphism set are both groups. The main result, Theorem 4.1 establishes the equivalence between categorical groups and *crossed modules* specified by pairs of groups (G, H). These results are known in the literature but we feel it is useful to present this coherent account, as there are many different conventions and definitions used in the literature and our presentation in this section provides us with notation, conventions, and results for use in later sections. We also include several examples here.
- Section 5 introduces the key notion of a *principal categorical bundle* $\mathbf{P} \to \mathbf{B}$, with a categorical group $\mathbf{G}$ as 'structure group' and with both $\mathbf{P}$ and $\mathbf{B}$ being categories. This general framework does not require $\mathbf{B}$ to have a smooth structure. We give examples, including one that uses backtrack-erased path spaces. We conclude the section by showing that the notion of 'reduction' of a principal bundle carries over to this categorical setting. (In this work we do not explore categorical analogs of local triviality, a topic that is central to most other works in this area.)
- In Section 6 we introduce the notion of a *decorated* principal bundle. This gives a useful example of a categorical principal bundle whose structure depends on the action of a given crossed module. Briefly, we start with an ordinary principal bundle $\pi : P \to B$ equipped with a connection $\overline{A}$; we form a categorical principal bundle whose base category $\mathbf{B}$ has object set B and backtrack-erased paths on B as morphisms; the bundle category $\mathbf{P}$ has object set P and morphisms of the form $(\tilde{\gamma}, h)$, with $\tilde{\gamma}$ being any $\overline{A}$-horizontal path on P and h running over a group H. Thus the morphisms are horizontal paths 'decorated' with elements of H. The result is a categorical principal bundle whose structure categorical group is specified by the pair of groups G and H, with G acting on objects of $\mathbf{P}$ and a semi-direct product of G and H acting on the morphisms of $\mathbf{P}$.
- In Section 7 we introduce the notion of a *categorical connection* on a categorical principal bundle. We present several examples, and then show, in Theorem 7.1, how to construct a categorical connection on the bundle of decorated paths and then, in Theorem 7.2, in a more abstractly decorated categorical bundle.
- In Section 8 we describe, in a precise way, categories whose morphisms are (equivalence classes of) paths. We do not use the popular practice of identifying thin-homotopy equivalent paths and explain how our approach provides a very convenient framework in which to formulate properties of parallel-transport such as invariance under reparametrizations, backtracks and thin-homotopies (Proposition 2.2, Theorems 3.1 and 3.2).
- In Section 9 we construct categorical connections at a 'higher' geometric level: here the objects are paths, and the morphisms are paths of paths.
- We present our final and most comprehensive example of a categorical connection in Section 10, where we develop parallel-transport of 'decorated' paths over a space of paths. Thus the transport of a decorated path $(\tilde{\gamma}, h)$ is specified through the data $(\tilde{\Gamma}, h, k)$, where $\tilde{\Gamma}$ is a path of paths on the bundle, horizontal with respect to a path space connection $\omega_{(A,B)}$, $h \in H$ decorates the initial (or source) path $\tilde{\gamma}_0 = s(\tilde{\Gamma})$ for $\tilde{\Gamma}$, and $k \in K$ encodes the rule for producing the resulting final decorated path $(\tilde{\gamma}_1, h_1)$.
- Section 11 presents a brief account of associated bundles, along with parallel-transport in such bundles, in the categorical framework.