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A new construction of Lagrangians in the complex Euclidean plane in terms of planar curves

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ABSTRACT

We introduce a new method to construct a large family of Lagrangian surfaces in complex Euclidean plane $\mathbb{C}^2$ by means of two planar curves making use of their usual product as complex functions and integrating the Hermitian product of their position and tangent vectors.

Among this family, we characterize minimal, constant mean curvature, Hamiltonian stationary, solitons for mean curvature flow and Willmore surfaces in terms of simple properties of the curvatures of the generating curves. As an application, we provide explicitly conformal parametrizations of known and new examples of these classes of Lagrangians in $\mathbb{C}^2$.

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1. Introduction

An isometric immersion $\phi : M^n \to \widetilde{M}^n$ of an n-dimensional Riemannian manifold M^n into an n-dimensional Kähler manifold $\widetilde{M}^n$ is said to be Lagrangian if the complex structure J of $\widetilde{M}^n$ interchanges each tangent space of M^n with its corresponding normal space. Lagrangian submanifolds appear naturally in several contexts of Mathematical Physics. For example, special Lagrangian submanifolds of the complex Euclidean space $\mathbb{C}^n$ (or of a Calabi–Yau manifold) have been studied widely and in [1] an explanation of mirror symmetry in terms of the moduli spaces of special Lagrangian submanifolds was proposed. These submanifolds are volume minimizing and, in particular, they are minimal submanifolds. In the two-dimensional case, special Lagrangian surfaces of $\mathbb{C}^2$ are exactly complex surfaces with respect to another complex structure on $\mathbb{R}^4 \equiv \mathbb{C}^2$.

The simplest examples of Lagrangian surfaces in $\mathbb{C}^2$ are given by the product of two planar curves $\alpha = \alpha(t)$, $t \in I_1 \subseteq \mathbb{R}$, and $\omega = \omega(s)$, $s \in I_2 \subseteq \mathbb{R}$:

$$(t, s) \overset{\phi}{\longmapsto} (\alpha(t), \omega(s)) . \tag{1.1}$$

Another fruitful method of construction of Lagrangian surfaces in $\mathbb{C}^2$ is obtained when one takes the particular version for the two-dimensional case of Proposition 3 in [2] (see also [3,4]), involving a planar curve $\alpha = \alpha(t)$, $t \in I_1 \subseteq \mathbb{R}$, and a

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Legendre curve $(\gamma_1, \gamma_2) = \gamma = \gamma(s), s \in I_2 \subseteq \mathbb{R}$, in the 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$

$$(t, s) \xmapsto{\phi} \alpha(t) \cdot \gamma(s) = \alpha(t)\,(\gamma_1(s), \gamma_2(s))\,. \tag{1.2}$$

In [5], it was presented a different method to construct a large family of Lagrangian surfaces in $\mathbb{C}^2$ using a Legendre curve $(\alpha_1, \alpha_2) = \alpha(t), t \in I_1 \subseteq \mathbb{R}$, in the anti De Sitter 3-space $\mathbb{H}^3_1 \subset \mathbb{C}^2$ and a Legendre curve $(\gamma_1, \gamma_2) = \gamma(s), s \in I_2 \subseteq \mathbb{R}$, in $\mathbb{S}^3 \subset \mathbb{C}^2$:

$$(t, s) \xmapsto{\phi} (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))\,. \tag{1.3}$$

We observe that in the constructions (1.1)–(1.3) the components of the position vector $\phi = (\phi_1, \phi_2)$ of the immersions are given by the product of two complex functions:

$$\phi_1(t, s) = \begin{cases} \alpha(t) \\ \alpha(t)\gamma_1(s) \\ \alpha_1(t)\gamma_1(s), \end{cases} \qquad \phi_2(t, s) \begin{cases} \omega(s) \\ \alpha(t)\gamma_2(s) \\ \alpha_2(t)\gamma_2(s). \end{cases} \tag{1.4}$$

From an algebraic point of view, we propose now to consider one of the components as a product of two complex functions and the other as an addition of another two complex functions. So, we can consider the following type of possible Lagrangian immersions:

$$\phi(t, s) = (f(t) + g(s), \alpha(t)\omega(s))\,, \tag{1.5}$$

where $\alpha = \alpha(t), t \in I_1 \subseteq \mathbb{R}$ and $\omega = \omega(s), s \in I_2 \subseteq \mathbb{R}$ are planar curves. If we impose that ϕ gives an orthonormal parametrization of a Lagrangian immersion, we have that $(\phi_t, \phi_s) = 0$, where $(\cdot, \cdot)$ denotes the usual bilinear Hermitian product of $\mathbb{C}^2$. Since $\phi_t = (f'(t), \alpha'(t)\omega(s))$ and $\phi_s = (\dot g(s), \alpha(t)\dot\omega(s))$ where $'$ (resp. $\dot{}$) means derivate with respect to t (resp. to s), we get

$$f'(t)\overline{\dot g(s)} + \alpha'(t)\overline{\alpha(t)}\omega(s)\overline{\dot\omega(s)} = 0. \tag{1.6}$$

So, essentially we can take

$$f(t) = -\int \alpha'(t)\overline{\alpha(t)}dt, \qquad g(s) = \int \dot\omega(s)\overline{\omega(s)}ds. \tag{1.7}$$

Putting this in (1.5) we can check that

$$\phi(t, s) = \left(\int \dot\omega(s)\overline{\omega(s)}ds - \int \alpha'(t)\overline{\alpha(t)}dt, \alpha(t)\omega(s)\right) \tag{1.8}$$

is a Lagrangian immersion constructed from two planar curves (see Theorem 2.1), well defined up to a translation in $\mathbb{C}^2$.

An interesting problem in this setting is to find non-trivial examples of Lagrangian surfaces with some given geometric properties. In this paper we pay our attention to an extrinsic point of view and focus on several classical equations involving the mean curvature vector and natural associated variational problems. In this way, we determine in our construction of Lagrangians not only those which are minimal, have parallel mean curvature vector or constant mean curvature, but also those ones that are Hamiltonian stationary, solitons of mean curvature flow or Willmore.

When we involve lines and circles in (1.8) we get the most regular surfaces: special Lagrangians (Corollary 3.1) and Hamiltonian stationary Lagrangians (Corollary 3.3). In this setting, we provide explicit conformal parametrizations of some known examples in terms of elementary functions and obtain some new examples of interesting Hamiltonian stationary Lagrangians. With some more sophisticated curves, we obtain a very large family of new Lagrangians with constant mean curvature vector (Corollary 3.4), which includes a (branched) Lagrangian torus. Our construction (1.8) is actually inspired by the Lagrangian translating solitons obtained in [6] associated to certain special solutions of the curve shortening flow that we recover in Corollary 3.6. We also provide Willmore Lagrangians when we consider free elastic curves (Corollary 3.7). Finally, we illustrate in Section 3.8 that we can also arrive at Lagrangian tori starting from certain closed curves.

The key point of the proof of all our results is the simple expression (2.5) for the mean curvature vector of the Lagrangian immersion in terms of the curvature functions of the generatrix curves.

2. The construction

In the complex Euclidean plane $\mathbb{C}^2$ we consider the bilinear Hermitian product defined by

$$(z, w) = z_1\bar w_1 + z_2\bar w_2, \quad z, w \in \mathbb{C}^2.$$

Then $\langle\, , \,\rangle = \mathrm{Re}(\, , \,)$ is the Euclidean metric on $\mathbb{C}^2$ and $\omega = -\mathrm{Im}(\, , \,)$ is the Kähler two-form given by $\omega(\,\cdot\, , \,\cdot\,) = \langle J\cdot, \cdot\rangle$, where J is the complex structure on $\mathbb{C}^2$.

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