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Quantum free Yang–Mills on the plane

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ABSTRACT

We construct a free-probability quantum Yang–Mills theory on the two dimensional plane, determine the Wilson loop expectation values, and show that this theory is the $N = \infty$ limit of $U(N)$ quantum Yang–Mills theory on the plane. Our model provides an example of a stochastic geometry, motivated by quantum field theory, based on free probability theory.

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1. Introduction

In this paper we use free probability and free stochastic calculus to construct a large-N limit of $U(N)$ quantum Yang–Mills theory on the Euclidean plane $\mathbb{R}^2$. While it has long been expected that free probability should play a central role in describing large-N limits of certain matrix-model quantum theories (see, for instance, Douglas [1] and Gopakumar and Gross [2]), the model we develop, along with the earlier work of Xu [3] in this context, may be the first concrete rigorously developed example of a free-probability based geometric quantum field theory.

Pure quantum Yang–Mills theory, with gauge group $U(N)$ and spacetime $\mathbb{R}^2$, is described by the Yang–Mills measure μ_g which is formally a measure on gauge equivalence classes of connections A and has formal density $e^{-\frac{1}{2g^2}\|F^A\|^2_{L^2}}$, where F^A is the curvature of a connection form A and $g > 0$ a coupling constant. It has been known, at least since the work of 't Hooft [4], that this theory has a meaningful and conceptually useful limit as $N \to \infty$, holding g^2N fixed.

There is a vast body of works in the physics literature on the large-N limit of $U(N)$ gauge theories (early works include those of Kazakov and Kostov [5,6]). On the mathematical side, Singer [7] showed that a mathematically meaningful 'master field' exists as the large-N limit of two-dimensional $U(N)$ quantum Yang–Mills theory. Our free-probability Yang–Mills theory may be viewed as a realization of this master field theory. The large-N limit of Wilson loop expectations in quantum Yang–Mills on $\mathbb{R}^2$ was studied by Xu [3] using the known distributions of the $U(N)$ Wilson loop expectation values. For a brief review see [8].

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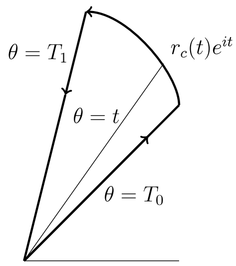

Fig. 1. A cross radial path.

2. Classical, quantum, and free

In making comparisons of our theory to classical differential geometric gauge theory, one should have in mind a $U(N)$ principal bundle over $\mathbb{R}^2$. For convenience we may trivialize the bundle and take the space $\mathcal{A}$ of all connections as the space of smooth $u(N)$-valued 1-forms on $\mathbb{R}^2$, where $u(N)$ is the Lie algebra of skew-Hermitian $N \times N$ matrices. The group $\mathcal{G}$ of gauge transformations consists of smooth maps $\phi : \mathbb{R}^2 \to U(N)$ and acts on $\mathcal{A}$ by

$$A^{\phi} = \phi A \phi^{-1} + \phi d\phi^{-1}.$$

It is usually more convenient to work with $\mathcal{G}_o$, the subgroup of $\mathcal{G}$ consisting of transformations which are the identity over the basepoint $o = (0, 0)$. Parallel transport by A along a piecewise smooth path $c : [T_0, T_1] \to \mathbb{R}^2$ is given by $h_c(T_1)$ where $h_c : [T_0, T_1] \to U(N)$ solves

$$dh_c(t) = -A\big(c'(t)\big)h_c(t)\,dt \quad \text{and} \quad h_c(T_0) = I,$$

with t running over $[T_0, T_1]$. For any $A \in \mathcal{A}$ there is a $\phi \in \mathcal{G}_o$ for which $A_{\mathrm{rad}} = A^{\phi}$ vanishes on radial vectors; this is radial gauge fixing, and identifies $\mathcal{A}/\mathcal{G}_o$ with the linear space $\mathcal{A}_o$ of $u(N)$-valued functions on $\mathbb{R}^2$ by associating each A to $f^{A_{\mathrm{rad}}}$ times the area 2-form on $\mathbb{R}^2$. The equation of parallel transport then associates to each $f \in \mathcal{A}_o$ and path $c : [T_0, T_1] \to \mathbb{R}^2 : t \mapsto r_c(t)e^{it}$ (see Fig. 1), the differential equation

$$dh_c(t) = i dM_c^f(t)h_c(t) \quad \text{and} \quad h_c(T_0) = I, \tag{2.1}$$

where $-iM_c^f(t)$ is the integral of f times area-form over the cone

$$S_c(t) = \{re^{i\theta} : \theta \in [T_0, t], 0 \le r \le r_c(\theta)\}. \tag{2.2}$$

Primarily we are interested in loops c based at the origin o, and then $h_c(T_1)$ is the classical *holonomy* of the connection on the loop c.

For quantum Yang–Mills theory with gauge group $U(N)$ the Yang–Mills measure is a probability measure specified formally by the expression

$$d\mu_g(A) = \frac{1}{Z_g} e^{-\frac{1}{2g^2}\|F^A\|^2_{L^2}}\,[DA],$$

where $F^A = dA + A \wedge A$ is the curvature of a connection form A, g is a parameter that may be viewed as a 'coupling constant', and $[DA]$ is formal Lebesgue measure on $\mathcal{A}_o$. Then μ_g is realized rigorously as Gaussian measure

$$d\mu_g(f) = \frac{1}{Z_g} e^{-\frac{1}{2g^2}\|f\|^2_{L^2}}\,Df \tag{2.3}$$

on a suitable completion of $L^2(\mathbb{R}^2) \otimes u(N)$. This will be reviewed in more detail below in Section 5, but for the sake of motivation for the free theory let us take a very quick look at the framework of the $U(N)$ theory. To each $f \in L^2(\mathbb{R}^2)$ is associated a $u(N)$-valued random variable $ib_{N,f}$, where $b_{N,f}$ is an $N \times N$ Gaussian Hermitian random matrix with mean 0 and entries having variances determined by $\frac{1}{N}\|f\|^2_{L^2}$. Following an idea of Gross [9], the classical differential Eq. (2.1) is replaced by the Itô stochastic differential equation

$$dh_c(t) = i dM_{N,c}(t)h_c(t) - \frac{1}{2} dM_{N,c}(t)^2 h_c(t) \quad \text{and} \quad h_c(T_0) = I, \tag{2.4}$$

where now $M_{N,c}(t)$ is $b_{N,f}$ with $f = 1_{S_c(t)}$. These equations were studied and the full theory of the quantum Yang–Mills measure on the plane were developed by Gross et al. [9] and Driver [10].

The large-N limit of the white noise process $f \mapsto b_{N,f}$, as $N \to \infty$, can be described by using free probability [11]. We proceed now to outline this framework very briefly and also describe the corresponding parallel transport process.

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