Journal of Geometry and Physics 62 (2012) 368–380

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On complex Landsberg and Berwald spaces

Nicoleta Aldea, Gheorghe Munteanu*

Transilvania University, Faculty of Mathematics and Informatics, Iuliu Maniu 50, Braşov 500091, Romania

ARTICLE INFO

Article history:
Received 28 November 2010
Accepted 22 October 2011
Available online 6 November 2011

MSC:
53B40
53C60

Keywords:
Complex Landsberg space
G-Landsberg space
Strong-Landsberg space
Complex Berwald space
Generalized Berwald space

ABSTRACT

In this paper, we study complex Landsberg spaces and some of their important subclasses. The tools of this study are the Chern–Finsler, Berwald, and Rund complex linear connections. We introduce and characterize the class of generalized Berwald and complex Landsberg spaces. The intersection of these spaces gives the so-called *G*-Landsberg class. This last class contains two other kinds of complex Finsler spaces: strong Landsberg and *G*-Kähler spaces. We prove that the class of *G*-Kähler spaces coincides with complex Berwald spaces, in Aikou's (1996) [1] sense, and it is a subclass of the strong Landsberg spaces. Some special complex Finsler spaces with (α, β)-metrics offer examples of generalized Berwald spaces. Complex Randers spaces with generalized Berwald and weakly Kähler properties are complex Berwald spaces.

1. Introduction

Real Landsberg spaces, in particular, real Berwald spaces, have been a major subject of study for many people over the years. In 1926, Berwald introduced a special class of Finsler spaces which took his name in 1964. It is known that a real Finsler space is called a Berwald space if the local coefficients of the Berwald connection depend only on position coordinates. An equivalent condition to this is that the Cartan tensor field is *h*-parallel to the Berwald connection, i.e., $C_{ijk;r} = 0$, where here ';' means the horizontal covariant derivative with respect to Berwald connection. In 1934, Cartan emphasized two weak points of the Berwald connection. One is that it is not metrical. Moreover, $g_{ij;k} = -2C_{ijk;0}$, and therefore, if $C_{ijk;0} = 0$, then it becomes metrical. However, such a space was called Landsberg by Berwald in 1928.

Many great contributions to the geometry of the real Landsberg and Berwald spaces are due to Szabo [2], Matsumoto [3], Antonelli [4], Bejancu [5], and Shen [6]. Every Berwald space is a Landsberg space. Proving the converse has been a long-standing problem [7–9].

Part of the general themes from real Finsler geometry about Landsberg and Berwald spaces [2–17] can be broached in complex Finsler geometry. However, there are sensitive differences compared to real reasonings, mainly on account of the fact that in complex Finsler geometry there exist two different covariant derivatives for the Cartan tensors, $C_{i\bar{j}k|h}$ and $C_{i\bar{j}k|\bar{h}}$. Such reason caused Aikou [1] to request in the definition of a complex Berwald space, beside the natural condition $C_{i\bar{j}k|h} = 0$, the Kähler condition.

Therefore, the same arguments will be taken into account in the definition of a complex Landsberg space. Using some ideas from the real case, related to the Rund and Berwald connections, our aim in the present paper is to introduce and study complex Landsberg spaces and some of their subclasses.

* Corresponding author. Tel.: +40 0268 276 567.
E-mail addresses: nicoleta.aldea@lycos.com (N. Aldea), gh.munteanu@unitbv.ro (G. Munteanu).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2011.10.010

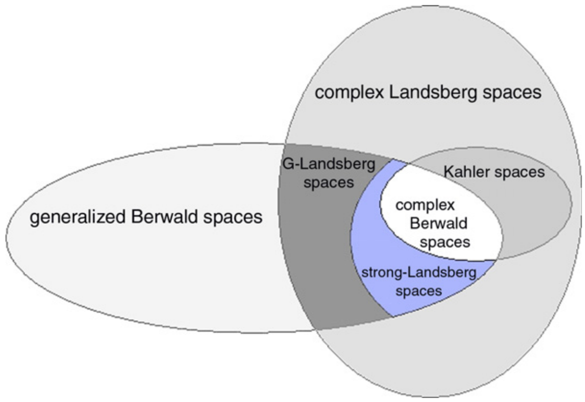

Fig. 1. Inclusions.

We associate to the canonical nonlinear connection, with the local coefficients $\overset{c}{N}{}^i_j := \dot{\partial}_j G^i$, two complex linear connections: one of Berwald type, $B\Gamma := (\overset{c}{N}{}^i_j, \overset{B}{L}{}^i_{jk}, \overset{B}{L}{}^i_{j\bar{k}}, 0, 0)$, and the other of Rund type, $R\Gamma := (\overset{c}{N}{}^i_j, \overset{c}{L}{}^i_{jk}, \overset{c}{L}{}^i_{j\bar{k}}, 0, 0)$. In the real case, a Finsler space is Landsberg if the Berwald and Rund connections coincide. But, in complex Finsler geometry, things are considerably more difficult, because in general the $B\Gamma$ and $R\Gamma$ connections are not of $(1, 0)$-type. Moreover, in the complex case, alongside the horizontal covariant derivative, with respect to $B\Gamma$ connection, we have its conjugate. Here, we speak of a complex *Landsberg* space iff $\overset{B}{L}{}^i_{jk} = \overset{c}{L}{}^i_{jk}$, and various characterizations of Landsberg spaces are proved in Theorem 3.1. Further on, we define the class of *G-Landsberg* spaces. It is in the class of complex Landsberg spaces with $\dot{\partial}_{\bar{k}} G^i = 0$. Theorem 3.2 reports on the necessary and sufficient conditions for a complex Finsler space to be a G-Landsberg space. A reinforcement of the tensorial characterization for a G-Landsberg space gives rise to a subclass of G-Landsberg, namely *strong Landsberg* iff $C_{l\bar{r}h\overset{B}{|}0} = 0$ and $C_{j\bar{r}h\overset{B}{|}\bar{0}} = 0$. Other important properties of strong Landsberg spaces are contained in Theorem 3.3.

Because any Kähler space is a complex Landsberg space, the substitution of the Landsberg condition with the Kähler condition in the definition of G-Landsberg spaces leads to another subclass of this, called *G-Kähler*. Inter alia, in Theorem 3.4, we prove that it coincides with the category of complex Berwald spaces defined by Aikou in [1]. The strong Landsberg spaces are situated somewhere between complex Berwald spaces and G-Landsberg spaces.

Complex Berwald spaces were introduced as a generalization of the real case, but in the particular context of Kähler. Therefore an unquestionable extension of these, directly related to the $B\Gamma$ connection, is called by us a *generalized Berwald* space. It has the coefficients $\overset{B}{L}{}^i_{jk}$ depending only on the position z. We give some characterizations for a generalized Berwald space; see Theorem 3.6.

An intuitive scheme with the introduced classes of complex Finsler spaces is given in Fig. 1.

The general theory on generalized Berwald spaces is completed by some special outcomes for the class of complex Finsler spaces with (α, β)-metrics. We prove that the complex Randers spaces under assumptions of generalized Berwald and weakly Kähler are complex Berwald; see Theorem 4.3. A class of complex Kropina spaces which are generalized Berwald is distinguished in Theorem 4.4.

The organization of the paper is as follows. In Section 2, we recall some preliminary properties of the n-dimensional complex Finsler spaces, together with some others needed for our aforementioned study. In Section 3, we prove the above-mentioned theorems, and we establish interrelations among all classes of complex Finsler spaces. In section Section 4, we produce some families of complex Finsler spaces with (α, β)-metrics which are generalized Berwald spaces and, in particular, complex Berwald spaces.

2. Preliminaries

In this section, we will give some preliminaries about complex Finsler geometry with Chern–Finsler, Berwald, and Rund complex linear connections. We will set the basic notions (for more see [10,18–21]) and we will prove some important properties of these connections.

2.1. Complex Finsler spaces

Let M be an n-dimensional complex manifold, and $z = (z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart.