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Berezin integration on non-compact supermanifolds

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ABSTRACT

We investigate the Berezin integral of non-compactly supported quantities. In the framework of supermanifolds with corners, we give a general, explicit and coordinate-free representation of the boundary terms introduced by an arbitrary change of variables. As a corollary, a general Stokes's theorem is derived—here, the boundary integral contains transversal derivatives of arbitrarily high order.

1. Introduction

Supermanifolds were introduced by Berezin, Leites and Konstant in the 1970s as a mathematical framework for the quantum theory of commuting and anticommuting fields. A remarkable contribution was Berezin's definition of his integral, in Ref. [1], predating the definition of supermanifolds by several years, and providing at the time sufficient indication that a reasonable supersymmetric analysis should exist.

Despite its utility, the integral suffers from a fundamental pathology: only the integral of *compactly supported* quantities is well-defined in a coordinate independent form—changes of variables introduce, in general, the so-called *boundary terms*. This can be seen as a major obstacle in the development of global superanalysis.

For example, although Stokes's theorem

$$\int_M d\omega = \int_{\partial M} \omega \tag{1}$$

has been extended to supermanifolds by Bernstein and Leites [2], this extension supposes that the supermanifold structure on the boundary ∂M enjoys a rather strong compatibility requirement. In fact, even for compactly supported integrands ω, the conclusion of the theorem *fails* in general, unless this assumption is made (*v.* Example 3.9).

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An invariant definition of the integral can however be made, on the basis of the following simple observation: For any supermanifold M, there exist morphisms $\gamma : M \to M_0$ – called *retractions* – which are left inverse to the canonical embedding $j_M : M_0 \to M$. Any retraction γ is a submersion whose fibres have compact base; thus, there is a well-defined *fibre integral* $\gamma_!$ which takes Berezin forms on M to volume forms on M_0, and one may define

$$\int_{(M,\gamma)} \omega = \int_{M_0} \gamma_!(\omega). \tag{2}$$

Taking pullback retractions, this definition is now trivially well-defined under coordinate changes. Furthermore, whereas retractions are non-unique in general, for certain classes of supermanifolds – *e.g.*, Lie supergroups G, homogeneous G-supermanifolds, and superdomains – there exist canonical retractions.

This framework allows us to give an explicit description of the behaviour of the integral under coordinate changes. To state our main result (Theorem 5.15), let $N \subset M^{p|q}$ be an open subspace of a supermanifold whose underlying space $N_0 \subset M_0$ is a manifold with corners. That is, we have $N_0 = \{\rho_i > 0 \mid i = 1, \dots, n\}$ for some functions ρ_i which define *boundary manifolds* $H_0 = \{\rho_{i_1} = \cdots = \rho_{i_k} = 0, \rho_j > 0\ (j \neq i_m)\}$. Let γ, γ' be retractions on N. On each H_0, one considers the supermanifold structure H induced by $\gamma^*(\rho_{i_m})$ and the retraction γ_H induced by γ. Let D_i be even vector fields such that $D_i(\gamma^*(\rho_j)) = \delta_{ij}$ on suitable neighbourhoods of $\{\gamma^*(\rho_i) = \gamma^*(\rho_j) = 0\}$.

Then, for any Berezin density ω such that the integrals exist,

$$\int_{(N,\gamma')} \omega = \int_{(N,\gamma)} \omega + \sum_{H \in B(\gamma^*(\rho))} \sum_{j \in J_H} \pm \int_{(H,\gamma_H)} \left(\omega_j . D^{j\downarrow}\right)|_{H,\gamma^*(\rho)}.$$

Here, we sum over all $H = \{\gamma^*(\rho_{i_1}) = \cdots = \gamma^*(\rho_{i_k}) = 0\}$ and all multi-indices $j \in J_H = \mathbb{N}^{\{i_1,\dots,i_k\}}$; moreover, $\omega_j := \frac{1}{j!}\left(\gamma'^*(\rho) - \gamma^*(\rho)\right)^j \omega$ and $j\downarrow$ denotes the multi-index j with entries reduced by one. The differential operators on the right hand side are of degree up to $\frac{q}{2} - 1$.

From this change of variables formula, we deduce a version of Stokes's theorem which is valid for an arbitrary supermanifold structure on the boundary (Corollary 5.21). Compared to Eq. (1), the right hand side depends not only on $\omega|_{\partial M}$, but on transversal derivatives up to order $\frac{q}{2} - 1$.

The question of defining the integral of non-compactly supported Berezinians was first studied by Rothstein in his seminal paper [3]. His fundamental insight was that the integral becomes well-defined if instead of the Berezinian sheaf, one considers the sheaf of super-differential operators with values in volume forms. This insight is vital—indeed, Rothstein's techniques form the basis of our investigations, and one may view Eq. (2) as an attempt to translate Rothstein's definition of the Berezin integral *via* the 'Fermi integral' to the realm of ordinary Berezinians.

For applications to superanalysis, Rothstein's sheaf is somewhat unwieldy, since it is an $\mathcal{O}_M$-module of infinite rank. For example, in the context of homogeneous supermanifolds, one frequently fixes integrands by invariance. Of course, this can only be done for $\mathcal{O}_M$-modules of rank one, which favours the Berezinian sheaf as a tool for superanalysis.

Our results have immediate applications to spherical harmonic analysis on Riemannian symmetric supermanifolds, in particular, the study of orbital and Eisenstein integrals in the spirit of Harish-Chandra, *cf.* Ref. [4]. Besides its relation to representation theory [5], this subject is of high current interest in mathematical physics, in the study of σ-model approximations of invariant random matrix ensembles, as are applied to disordered metals and topological insulators [6–9].

Let us end with a brief synopsis of our paper. In Section 2, we recall some basic facts and define the integral of Berezin densities with respect to a retraction. In Section 3, we prove a version of Stokes's theorem in this setting (Theorem 3.8). Here, the supermanifold structure on the boundary has to be chosen compatibly (see below). In Section 4, we prove a version of our change of variables formula in terms of coordinates (Theorem 4.5). Here, the 'boundary' nature of the 'boundary terms' is not yet evident. This is finally accomplished in Section 5, where the language and technique of supermanifolds with corners and boundary supermanifolds is introduced; here, the point of view of retractions proves particularly fruitful. By applying this machinery, we prove our main result (Theorem 5.15) and illustrate its use in some examples. Finally, we deduce a generalised Stokes's theorem (Corollary 5.21) where the supermanifold structure on the boundary is arbitrary.

2. The Berezin integral in the non-compact case

We use the standard definition of supermanifolds in terms of ringed spaces. For basic facts on these, we refer the reader to [10,11]. Let us fix our notation. Given an object in the graded category, we will denote the underlying ungraded object by a subscript 0. We denote supermanifolds as $M = (M_0, \mathcal{O}_M)$, $N = (N_0, \mathcal{O}_N)$, *etc.* Unless the contrary is stated explicitly, we will assume M, N to be of dimension (p, q). Manifolds will always be Hausdorff and second countable. By writing $U \subseteq M$ we will mean that U is the ringed subspace $M|_{U_0} := (U_0, \mathcal{O}_M|_{U_0})$ of M given by the open subset $U_0 \subseteq M_0$. Thus, unions and finite intersections of open subspaces are defined. Further, the set of superfunctions $\mathcal{O}_M(U_0)$ on U is abbreviated by $\mathcal{O}(U)$. Morphisms of supermanifolds $M \to N$ are denoted $\varphi = (\varphi_0, \varphi^*)$, with underlying smooth map $\varphi_0 : M_0 \to N_0$ and the sheaf morphism $\varphi^* : \mathcal{O}_N \to \varphi_{0*}\mathcal{O}_M$. For a given supermanifold M we denote the canonical embedding by $j_M : M_0 \to M$. Given $f \in \mathcal{O}(U)$, we write f_0 for $j_M^*(f)$.