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q-Deformed quantum Lie algebras

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Abstract

Attention is focused on q-deformed quantum algebras with physical importance, i.e. $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra. The main concern of this article is to assemble important ideas about these symmetry algebras in a consistent framework which will serve as starting point for representation theoretic investigations in physics, especially quantum field theory. In each case considerations start from a realization of symmetry generators within the differential algebra. Formulae for coproducts and antipodes on symmetry generators are listed. The action of symmetry generators in terms of their Hopf structure is taken as the q-analog of classical commutators and written out explicitly. Spinor and vector representations of symmetry generators are calculated. A review of the commutation relations between symmetry generators and components of a spinor or vector operator is given. Relations for the corresponding quantum Lie algebras are computed. Their Casimir operators are written down in a form similar to that for the undeformed case.

Keywords: Quantum groups; q-Deformation; Quantum Lie algebras; Non-commutative geometry; Spacetime symmetries

1. Introduction

It is an old idea that limiting the precision of position measurements by a fundamental length will lead to a new method for regularizing quantum field theories [20]. It is also well known that

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such a modification of classical spacetime will in general break its Poincaré symmetry [45]. One way out of this difficulty is to change not only spacetime, but also its underlying symmetry.

Quantum groups can be seen as deformations of classical spacetime symmetries, as they describe the symmetry of their comodules, the so-called quantum spaces. From a physical point of view the most realistic examples for quantum groups and quantum spaces arise from q-deformation [14,15,21,24,33,39,54]. In our work we are interested in q-deformed versions of Minkowski space and Euclidean spaces as well as their corresponding symmetries, given by q-deformed Lorentz algebra and algebras of q-deformed angular momentum, respectively [6,26,29,38,42]. Remarkably, Julius Wess and his coworkers were able to show that q-deformation of spaces and symmetries can indeed lead to discretizations, as they result from the existence of a smallest distance [9,16]. This observation nourishes the hope that q-deformation might give a new method for regularizing quantum field theories [5,19,28,36].

In our previous work [1,34,43,48–51] attention was focused on q-deformed quantum spaces of physical importance, i.e. the two-dimensional Manin plane, q-deformed Euclidean space in three or four dimensions and q-deformed Minkowski space. If we want to describe fields on q-deformed quantum spaces we need to consider representations of the corresponding quantum symmetries, given by $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra. The study of such quantum algebras has produced a number of remarkable results during the last two decades. For a review we recommend to the reader the presentations in [11,22,32] and references therein. In this article we want to adapt these general ideas to our previous considerations about q-deformed quantum spaces. In doing so, we provide a basis for performing concrete calculations, as they are necessary in formulating and evaluating field theories on quantum spaces.

In particular, we intend to proceed as follows. In Section 2 we cover the ideas that our considerations about q-deformed quantum symmetries are based on. In the subsequent sections we first recall for each quantum algebra under consideration how its generators are realized within the corresponding q-deformed differential calculus. Then we are going to present explicit formulae for coproduct and antipode on a set of independent symmetry generators. With this knowledge at hand we should be able to write down explicit formulae for so-called q-commutators between symmetry generators and representation space elements. In addition to this, we are going to consider spinor and vector representations of the independent symmetry generators and give a complete review of the commutation relations between symmetry generators and components of a spinor or vector operator. Furthermore we are going to calculate the adjoint action of the independent symmetry generators on each other. In this manner, we will get relations for quantum Lie algebras. We will close our considerations by writing down q-analogs of Casimir operators. Finally, Section 6 will serve as a short conclusion. For reference and for the purpose of introducing consistent and convenient notation, we provide a review of key notation and results in Appendix A.

We should also mention that most of our results were obtained by applying the computer algebra system Mathematica [53]. We are convinced that in the future the use of this powerful tool will be inevitable in managing the extraordinary complexity of q-deformation.

2. Basic ideas on q-deformed quantum symmetries

Roughly speaking a quantum space is nothing else than an algebra generated by non-commuting coordinates $X_1, X_2, \ldots, X_n$, i.e.

$$\mathcal{A}_q = \mathbb{C}[[X_1, \ldots, X_n]]/\mathcal{I}, \tag{1}$$