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Review

Geometry of jet spaces and integrable systems

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A B S T R A C T

An overview of some recent results on the geometry of partial differential equations in application to integrable systems is given. Lagrangian and Hamiltonian formalism both in the free case (on the space of infinite jets) and with constraints (on a PDE) are discussed. Analogs of tangent and cotangent bundles to a differential equation are introduced and the variational Schouten bracket is defined. General theoretical constructions are illustrated by a series of examples.

Contents

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0. Introduction

The main task of this paper is to overview a series of our results achieved recently in understanding integrability properties of partial differential equations (PDEs) arising in mathematical physics and geometry [1–10]. These results are essentially based on the geometrical approach to PDEs developed since the 1970s by Vinogradov and his school (see [11–15] and references therein). The approach treats a PDE as an (infinite-dimensional) submanifold in the space $J^\infty(\pi)$ of infinite jets for a bundle $\pi : E \to M$ whose sections play the role of unknown functions (fields). This attitude allowed to apply to PDEs powerful techniques of differential geometry and homological algebra. The latter, in particular, made it possible to give an invariant and efficient formulation of higher-order Lagrangian formalism with constraints and calculus of variations (see [16,17,14,18], see also [19–25]) and, as it became clear later, was a bridge to BRST cohomology in gauge theories, anti-field formalism and related topics, [26]; see also [27–29].

Geometrical treatment of differential equations has a long history and originates in the works by Lie [30–32], as well as in research by Bäcklund [33], Monge [34], Darboux [35], Bianchi [36] and, later, by Cartan [37]. Note incidentally that Cartan's theory of involutivity for external differential systems was an inspiration for another cohomological theory associated to PDEs and developed in papers by Spencer and his school, [38,39]. Spencer's work, (the so-called formal theory), closely relates to earlier and unfairly forgotten results by Janet [40] and Riquier [41]; see also [42] as well as [43–46].

A milestone in the geometry of differential equations was introduction by Charles Ehresmannn of the notion of jet bundles [47,48] that became the most adequate language for Lie's theory, but a real revival of the latter came with the works by Ovsyannikov (see his book [49] on group analysis of PDEs; see also [50–53]).

A new impulse for the reappraisal of the Sophus Lie heritage was given by the discovery of integrability phenomena in nonlinear systems [54–56,138,57–61] near the end of the 1960s[1] and Hamiltonian interpretation of this integrability [65,57]. In particular, it became clear that integrable equations possess infinite series of higher, or generalised, symmetries (see [66,11]), and classification of evolution equations with respect to this property allowed to discover new, at that time, integrable equations, [67–69]. Later the notion of a higher symmetry was generalised further to that of a nonlocal one [70] and the search for a geometrical background of nonlocality led to the concept of a differential covering [71]. The latter proved to play an important role in the geometry of PDEs and we discuss it in our review.

It also became clear that the majority of integrable evolutionary systems possess a bi-Hamiltonian structure [72,73], i.e., can be represented as Hamiltonian flows on the space of infinite jets in at least two different ways and the corresponding Hamiltonian structures are compatible. The bi-hamiltonian property, by Magri's scheme [73], leads to the existence of infinite series of commuting symmetries and conservation laws. In addition, it gives rise to a recursion operator for higher symmetries that is an efficient tool for the practical construction of symmetry hierarchies. Nevertheless, recursion operators exist for equations possessing no Hamiltonian structure at all (e.g., for the Burgers equation). A self-contained cohomological approach to recursion operators based on Nijenhuis brackets and related to the theory of deformations for PDE structures is exposed in [15].

The literature on the Hamiltonian theory of PDEs is vast and we confine ourselves here to the key references [74–77], but one feature is common to all research: theories and techniques are applicable to evolution equations only. Then a natural question arises: what to do if the equation at hand is not represented in the evolutionary form? We believe that (at least, a partial) answer to this question can be found in this paper.

Of course, one of possible solutions is to transform the equation to the evolutionary form. However:

- Not all equations can be rendered to this form.[2]
- How to check independence of Hamiltonian (and other) structures on a particular representation of our equation? In other words, if we found a Hamiltonian operator in one representation, what guarantees that it survives when the representation is changed?
- Even if the answer to the previous question is positive, how are the results transformed when passing to the initial form of the equation?

In what follows, we treat any concrete equation "as is" and try to uncover those objects and constructions that are naturally associated to this equation. In particular, we do not assume existence of any additional structures that enrich the equation. Such structures are by all means extremely interesting and lead to very nontrivial classes of equations (e.g., equations of hydrodynamical type [78], Monge–Ampére equations [79] or equations associated to Lie groups [80]), but here we look for internal properties of an arbitrary PDE.

As said in the very beginning, an equation (or, to be more precise, its infinite prolongation, i.e., equation itself together with all its differential consequences) is a submanifold in a jet space $J^\infty(\pi)$. To escape technical difficulties, we consider the simplest case, when $\pi : E \to M$ is a locally trivial vector bundle, though all the results remain valid in a more complicated situation (e.g., for jets of submanifolds). The reader who is interested in local results only may keep in mind the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ instead of π.

1 Though discussions on "what is integrability" continued [62] and are still held now (see, e.g. quite recent papers [63] or [64]).

2 For example, gauge invariant equations, such the Yang–Mills, Maxwell, Einstein equation, etc., can not be presented in the evolutionary form.