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On drift, diffusion and geometry

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Abstract

We present some reflections on the links between drift, diffusion and geometry. For this purpose, we examine different sources of "diffusion models", in physics and in mathematics. We observe that diffusion processes may arise from original models either deterministic, or random but where dynamics and noise are clearly delineated. In the end, we get a diffusion process where noise and dynamics ("drift") are generally intimately entangled in a second-order partial differential operator. We focus on the following questions. Are there implicit geometric structures to properly define a diffusion? How are drift/dynamics and diffusion mixed? Are there geometric structures needed to separate drift and diffusion? We stress the importance of recurrent differential geometric structures – connections and Riemannian metrics – needed to properly define a "diffusion term" and also to separate drift from diffusion.

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1. Introduction

Consider a set of particles in movement. One may say that drift is a general trend followed by these particles, while diffusion is random wandering. Going beyond words, there are different mathematical objects to capture the ideas of drift and diffusion.

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Concerning diffusion, Laplacian Δ and Brownian motion are intimately related. The Brownian motion may be seen as the prototype of continuous time noise. This diffusion process shares, with its infinitesimal generator $\frac{1}{2}\Delta$, strong spatial symmetries as the invariance by the Euclidian group. It is a sort of "pure noise" on $\mathbb{R}^n$, in the sense that no direction is privileged (isotropy, absence of drift). The drift is generally a vector field, giving direction to the solutions of an ordinary differential equation or driving the partial differential equation (PDE) equation for the density.

We shall make a brief review of different sources of "diffusion models". We shall focus on the following questions:

(1) are there implicit geometric structures to properly define a diffusion?
(2) how are drift and diffusion mixed?
(3) are there geometric structures needed to separate drift and diffusion?

We shall see that diffusion processes may arise from original models either deterministic, or random but where dynamics and noise are clearly delineated. In the end, we get a diffusion process where noise and dynamics ("drift") are generally intimately entangled, either in a second-order partial differential operator, or in a stochastic differential equation. We shall question this linkage and examine the geometric structures that may be needed to properly define a "diffusion term" and also to separate drift from diffusion. We shall also observe recurrent differential geometric structures: connections, Riemannian metrics.

This paper originates from reflections following a series of papers [1–5]. Classically associating a Riemannian metric g to a nondegenerate elliptic operator L on a manifold $\mathbb{M}$ [6], we exploited geometric properties of the Riemannian manifold $(\mathbb{M}, \mathsf{g})$ to exhibit properties of the diffusion process with infinitesimal generator L, and study various types of problem (symmetries, finite dimensional filters, group invariant solutions). Starting from stochastic problems formulated in terms of diffusion and drift, we were thus led to geometric problems: therefore, our reflection focused on the link between drift/diffusion and geometry. This paper is a tentative review to clarify the question.

Section 2 collects the mathematical background which will appear recurrently. In Sections 3 and 4, we revisit two physical models of diffusion – Fick's law and deterministic interacting particles systems – and we focus on the role of underlying geometric structures. Then, in Section 5, we examine diffusion processes and stochastic differential equations under the same geometrical angle. After having seen examples of how noise and dynamics entangle and the role of geometric structures, we turn to the reverse problem in Section 6. We shall examine operations to disentangle noise and dynamics, and try to properly define the "drift" of a diffusion process. In conclusion, we sum up our observations and sketch some recurrent facts.

2. Mathematical background

We collect here the main mathematical background needed in the sequel. In what follows, $\mathbb{M}$ is a smooth manifold.