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Explicit series solution of travelling waves with a front of Fisher equation

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Abstract

In this paper, an analytic technique, namely the homotopy analysis method, is employed to solve the Fisher equation, which describes a family of travelling waves with a front. The explicit series solution for all possible wave speeds $0 < c < +\infty$ is given. Such kind of explicit series solution has never been reported, to the best of author's knowledge. Our series solution indicates that the solution contains an oscillation part when $0 < c < 2$. The proposed analytic approach is general, and can be applied to solve other similar nonlinear travelling wave problems.

1. Introduction

The nonlinear reaction–diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + vu(1-u), \tag{1}$$

was first introduced by Fisher as a model for the propagation of a mutant gene [1]. It has wide application in the fields of logistic population growth [2,3], flame propagation [4], neurophysiology [5], autocatalytic chemical reactions [6], branching Brownian motion processes [7], and nuclear reactor theory [8].

In chemical media the function $u(x,t)$ is the concentration of the reactant. D represents its diffusion coefficient, and the positive constant v specifies the rate of chemical reaction. In media of other natures, u might be temperature or electric potential, D might be the thermal conductivity or specific electrical conductivity. The medium described by Eq. (1) is often referred to as a bistable medium because it has two homogeneous stationary states, $u = 0$ and $u = 1$. A kink-like travelling wave solution of Eq. (1) describes a constant-velocity front of transition from one homogeneous state to another.

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Several analytic methods have been developed for obtaining travelling wave solutions: the inverse scattering transfer [9], the Hirota method [10], Lamb's ansatz [11,12], Adomian decomposition method [13], etc. [14]. The problem of obtaining solutions for systems including dissipative losses, e.g., reaction–diffusion systems, turned out to be complex. Most of the above-mentioned methods do not work [15]. The usual way of treating the problem is by perturbation theory or numerical investigation.

Writing $t^* = vt$, $x^* = (v/D)^{1/2}x$, and dropping the star, Eq. (1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2. \tag{2}$$

In the spatial homogeneous situation the steady states are $u = 0$ and $u = 1$, which are respectively unstable and stable. This suggests that we should look for travelling wave front solutions to Eq. (2).

Consider a travelling wave solution

$$u(x,t) = u(x - ct) = f(z), \quad z = x - ct, \tag{3}$$

propagating with a speed c. It obeys the ordinary differential equation

$$f'' + cf' + f - f^2 = 0, \tag{4}$$

subject to the boundary conditions

$$f(-\infty) = 1, \qquad f(+\infty) = 0, \tag{5}$$

where the prime denotes the differentiation with respect to z.

The above wave solutions are invariant to any shift in the origin of the coordinate system, because Eq. (4) and the boundary condition (5) are the same under the transformation $z = z + C$ for any a constant C. Moreover, Fisher [1] found that Eq. (4) has an infinite number of travelling wave solutions for which $0 \leqslant f \leqslant 1$ and $c > c_{\min} = 2$. A closed-form solution of travelling wave, i.e.,

$$f(z) = \frac{1}{\left[1 + e^{z/\sqrt{6}}\right]^2}, \tag{6}$$

was found by Ablowitz and Zeppetella [16] for a particular wave speed

$$c = 5/\sqrt{6} = 2.04124.$$

By means of shifting the origin, the expression (6) can be rewritten as

$$f(z) = \frac{1}{\left[1 + (\sqrt{2} - 1)e^{z/\sqrt{6}}\right]^2}, \tag{7}$$

corresponding to $f(0) = 1/2$. This kind of solution was found by many authors (for example, see also [17]). For large wave speed c, there is a small parameter $\varepsilon = 1/c^2 \ll 1$. In this case, Murray [18] applied perturbation method to obtain an asymptotic solution

$$f(z) = \frac{1}{1 + e^{z/c}} + e^{z/c}\frac{\ln[4e^{z/c}(1 + e^{z/c})^{-2}]}{c^2(1 + e^{z/c})^2} + O(c^{-4}), \tag{8}$$

which is however valid only for large wave speed c. Solutions for $c < 2$ are scarcely reported. To the best of our knowledge, analytic solutions for general wave speed c have been never reported.

The homotopy [19] is a basic concept in topology [20]. Based on the homotopy, some numerical techniques such as the continuation method [21] and the homotopy continuation method [22] were developed. There is a suite of FORTRAN subroutines in Netlib for numerically solving nonlinear systems of equations by homotopy methods, called HOMPACK. In 1992, a kind of analytic method, namely the homotopy analysis method (HAM) [23,24], was developed to solve highly nonlinear problems. Different from perturbation techniques [25], the homotopy analysis method does not depend upon any small or large parameters and thus is valid for most of nonlinear problems in science and engineering. Besides, it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [26], the δ-expansion method [27], and Adomian's decomposition method [28], as proved by Liao in his book [24]. Currently, Liao [29] showed that the so-called "homotopy perturbation method" (proposed by Dr. He in 1998) is only a special case of the homotopy analysis method. The homotopy analysis method has been successfully applied to many nonlinear problems [30–39]. In this paper, by means of the homotopy analysis method, the explicit series solution of Fisher equation for all possible wave speed $0 < c < +\infty$ is given.