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# Helmholtz conditions and symmetries for the time dependent case of the inverse problem of the calculus of variations

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ABSTRACT

We present a reformulation of the inverse problem of the calculus of variations for time dependent systems of second order ordinary differential equations using the Frölicher–Nijenhuis theory on the first jet bundle, $J^1\pi$. We prove that a system of time dependent SODE, identified with a semispray $S$, is Lagrangian if and only if a special class, $\Lambda^1_S(J^1\pi)$, of semi-basic 1-forms is not empty. We provide global Helmholtz conditions to characterize the class $\Lambda^1_S(J^1\pi)$ of semi-basic 1-forms. Each such class contains the Poincaré–Cartan 1-form of some Lagrangian function. We prove that if there exists a semi-basic 1-form in $\Lambda^1_S(J^1\pi)$, which is not a Poincaré–Cartan 1-form, then it determines a dual symmetry and a first integral of the given system of SODE.



## 1. Introduction

In this work we present a reformulation of the inverse problem for time dependent systems of second order ordinary differential equations in terms of semi-basic 1-forms. In this approach we solely make use of the Frölicher–Nijenhuis theory on $\pi_{10} : J^1\pi \to M$, the first jet bundle of an $(n+1)$-dimensional, real, smooth manifold $M$, which is fibred over $\mathbb{R}$, $\pi : M \to \mathbb{R}$. We characterize when the time dependent system of SODE

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + 2G^i\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0 \tag{1.1}$$

is equivalent to the system of Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad y^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}, \tag{1.2}$$

for some smooth Lagrangian function $L$ on $J^1\pi$, in terms of a class $\Lambda^1_S(J^1\pi)$ of semi-basic 1-forms on $J^1\pi$. This work is a natural extension of the time independent case studied in [1]. However, the time dependent framework has particular aspects that

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result in some differences of this approach from the one studied in [1]. Lagrangian systems of time independent differential equations are always conservative and this is not the case in the time dependent framework. In this approach we use the formalism developed for the inverse problem of the calculus of variations to search for symmetries as well. In other words, this approach gives the possibility of searching for dual symmetries for the time dependent system (1.1) of SODE in the considered class $\Lambda^1_S(J^1\pi)$ of semi-basic 1-forms.

Necessary and sufficient conditions under which the two systems (1.1) and (1.2) can be identified using a multiplier matrix are usually known as the *Helmholtz conditions*. This inverse problem was entirely solved only for the case $n = 1$ by Darboux, in 1894, and for $n = 2$ by Douglas, in 1941 [2]. A geometric reformulation of the Douglas approach, using linear connections arising from a system of SODE and its associated geometric structures, can be found in [3,4]. The relation between the inverse problem of the calculus of variations and the condition of self-adjointness for the equations of variation of the system (1.1) was studied by Davis in 1929, [5]. In 1935, Kosambi [6], has obtained necessary and sufficient conditions, that were called later Helmholtz conditions, for the equations of variations of the system (1.1) to be self-adjoint. For various approaches to derive the Helmholtz conditions in both the autonomous and nonautonomous case, we refer to [7–14]. See also [1] for a reformulation of the Helmholtz conditions in terms of semi-basic 1-forms in the time independent case.

In this paper we study the inverse problem of the calculus of variations when the time dependent system (1.1) of SODE is identified with a semispray $S$ on the first jet bundle $J^1\pi$. We seek for a solution of the inverse problem of the calculus of variations in terms of semi-basic 1-forms on $J^1\pi$. We first show, in Theorem 4.5, that a semispray $S$ is a Lagrangian vector field if and only if there exists a class of semi-basic 1-forms $\theta$ on $J^1\pi$ such that their Lie derivatives $\mathcal{L}_S\theta$ are closed 1-forms. We denote this class by $\Lambda^1_S(J^1\pi)$. These results reformulate, in terms of a semi-basic 1-form, the results expressed in terms of a 2-form, obtained in [13,15,8,16,11]. In Proposition 4.6 we strengthen the results of Theorem 4.5 and prove that, if nonempty, the set of semi-basic 1-forms $\Lambda^1_S(J^1\pi)$ always contains the Poincaré–Cartan 1-form of some Lagrangian function $L$. Moreover, we show that any semi-basic 1-form $\theta \in \Lambda^1_S(J^1\pi)$, which is not the Poincaré–Cartan 1-form of some Lagrangian function, determines a *first integral* and a *dual symmetry* of the Lagrangian system.

In Theorems 5.1 and 5.2 we characterize the semi-basic 1-forms of the set $\Lambda^1_S(J^1\pi)$, depending if they represent or not Poincaré–Cartan 1-forms. First, we pay attention to $\mathrm{d}_J$-closed, semi-basic 1-forms $\theta$, where $J$ is the vertical endomorphism. This class of semi-basic 1-forms $\theta$ coincides with the class of Poincaré–Cartan 1-forms corresponding to the Lagrangian function $L = i_S\theta$, see Lemma 4.2. For this class we prove, in Theorem 5.1, that the inverse problem has a solution if and only if the Poincaré–Cartan 1-form is $\mathrm{d}_h$-closed, where $h$ is the horizontal projector induced by the semispray.

In Theorem 5.2 we formulate a coordinate free version of the Helmholtz conditions in terms of semi-basic 1-forms on the first jet bundle $J^1\pi$. If there exists a semi-basic 1-form $\theta$ that satisfies the Helmholtz conditions, then $\theta$ is equivalent (modulo $\mathrm{d}t$) to the Poincaré–Cartan 1-form of some Lagrangian function. Moreover, if such $\theta$ is not $\mathrm{d}_J$-closed then $i_S\mathrm{d}\theta$ is a dual-symmetry and induces a first integral for the semispray $S$. To derive the Helmholtz conditions in Theorem 5.2 we make use of the Frölicher–Nijenhuis theory on $J^1\pi$ developed in Section 2.2, as well as of the geometric objects induced by a semispray that are presented in Section 3.

Since semi-basic 1-forms play a key role in our work, we pay a special attention to this topic in Section 2.3. In this section we prove a Poincaré-type lemma for the differential operator $\mathrm{d}_J$ restricted to the class of semi-basic 1-forms on $J^1\pi$. The Poincaré-type lemma will be very useful to characterize those semi-basic 1-forms that are the Poincaré–Cartan 1-forms of some Lagrangian functions.

An important tool in this work is discussed in Section 3.2, it is a tensor derivation on $J^1\pi$ that is called the dynamical covariant derivative induced by a semispray. This derivation has its origins in the work of Kosambi [6], where it has been introduced with the name of "bi-derivative". The notion of dynamical covariant derivative was introduced by Cariñena and Martinez in [17] as a covariant derivative along the bundle projection $\pi_{10}$. See also [18–22]. It can be defined also as a derivation on the total space $TJ^1\pi$, see [23,24].

## 2. Preliminaries

### 2.1. The first jet bundle $J^1\pi$

For a geometric study of time dependent systems of SODE the most suitable framework is the affine jet bundle $(J^1\pi, \pi_{10}, M)$, see [19,25,23,26]. We consider an $(n+1)$-dimensional, real and smooth manifold $M$, which is fibred over $\mathbb{R}$, $\pi : M \to \mathbb{R}$. The first jet bundle of $\pi$ is denoted by $\pi_{10} : J^1\pi \to M$, $\pi_{10}(j^1_t\gamma) = \gamma(t)$, for $\gamma$ a local section of $\pi$ and $j^1_t\gamma$ the first jet of $\gamma$ at $t$. A local coordinate system $(t, x^i)_{i\in\overline{1,n}}$ on $M$, where $t$ represents the global coordinate on $\mathbb{R}$ and $(x^i)$ the $\pi$-fiber coordinates, induces a local coordinate system on $J^1\pi$, denoted by $(t, x^i, y^i)$. Submersion $\pi_{10}$ induces a *natural foliation* on $J^1\pi$. Coordinates $(t, x^i)$ are transverse coordinates for this foliation, while $(y^i)$ are coordinates for the leaves of the foliation.

Throughout the paper we assume that all objects are $C^\infty$-smooth where defined. The ring of smooth functions on $J^1\pi$, the $C^\infty$ module of vector fields on $J^1\pi$ and the $C^\infty$ module of $k$-forms are respectively denoted by $C^\infty(J^1\pi)$, $\mathfrak{X}(J^1\pi)$ and $\Lambda^k(J^1\pi)$. The $C^\infty$ module of $(r, s)$-type tensor fields on $J^1\pi$ is denoted by $\mathcal{T}^r_s(J^1\pi)$ and $\mathcal{T}(J^1\pi)$ denotes the tensor algebra on $J^1\pi$. By a vector valued $l$-form ($l \geq 0$) on $J^1\pi$ we mean an $(1, l)$-type tensor field on $J^1\pi$ that is skew-symmetric in its $l$ arguments.