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Jordan–Hölder reductions for principal Higgs bundles on curves☆

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ABSTRACT

It is known that semistable sheaves V admit a filtration whose quotients are stable and have the same slope of V, named the Jordan–Hölder filtration. We give the analogous result for principal Higgs bundles on curves. Let G be a reductive algebraic group over $\mathbb{C}$, if $\mathfrak{E} = (E, \phi)$ is a semistable principal Higgs G-bundle, there exists a parabolic subgroup P of G and an admissible reduction of the structure group of $\mathfrak{E}$ to that parabolic such that the principal Higgs bundle obtained by extending the structure group to the Levi factor $L(P)$ of P is a stable principal Higgs bundle. The extension of the structure group $L(P)$ to G of the latter stable principal bundle is the graded module $\mathrm{gr}(\mathfrak{E})$.

1. Introduction

The background of this paper is the well-known Jordan–Hölder filtration for semistable coherent sheaves on a projective variety. If V is a semistable coherent sheaf on a projective variety, there exists a filtration by coherent sheaves whose quotients are stable and have the same slope as V. These quotients in general are not locally free, unless, of course, the variety is a smooth curve. In case the coherent sheaf is polystable, the graded module $\mathrm{gr}(V)$, the direct sum of the quotients of the above filtration, coincides with V. The filtration constructed as above is named Jordan–Hölder filtration.

In [1], Simpson proves the existence of the Jordan–Hölder filtration for semistable Higgs sheaves $\mathfrak{E}$. A Higgs sheaf $\mathfrak{E}$ is a pair (E, ϕ) where E is a sheaf and $\phi : E \to E \otimes \Omega^1$ a morphism such that $\phi \wedge \phi = 0$. A Higgs subsheaf of $\mathfrak{E}$ is a subsheaf $\mathfrak{F}$ such that $\phi(\mathfrak{F}) \subset \mathfrak{F} \otimes \Omega^1$. Moreover, the filtration is made up by Higgs subsheaves whose quotients are stable (in the sense of Higgs sheaves) and they have the same slope of $\mathfrak{E}$.

We may also consider the case of principal bundles. In [2], Ramanathan constructs the analogue to the Jordan–Hölder filtration for semistable principal bundles on curves. If $E \simeq Gl(V)$ is the bundle of linear frames of a semistable vector bundle on a projective curve, the structure group of E is the reductive linear group $Gl(n, \mathbb{C})$, and giving a filtration of V is the same as giving a reduction of the structure group of E to a parabolic subgroup P of $Gl(n, \mathbb{C})$. This means to obtain a principal P-bundle E_P on C together with an equivariant embedding $i : E_P \hookrightarrow E$ and implies that one is fixing a flag of subspaces of V invariant by the action of P. In some sense a principal bundle is a "locally free" object, and this is going to be a problem unless the base variety is a curve.

Moreover, the quotients of Jordan–Hölder filtration of a coherent sheaf E have the same slope as E. This condition for principal bundles translates into the condition that for any character $\chi : P \to \mathbb{C}^*$ which is trivial on the radical of G, the line bundle $E_P(\chi) \simeq E_P \times_\chi \mathbb{C}$ on C has degree zero.

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If E is a principal bundle on a curve with reductive structure group G, the Jordan–Hölder filtration is a reduction of E to a parabolic subgroup P, with the additional condition for characters of P being zero on the radical of G.

This paper contains the construction of the latter filtration for principal Higgs bundles. A principal Higgs bundle is a pair (E, ϕ) where E is a principal bundle and $\phi \in \Gamma(\mathrm{Ad}(E) \otimes \Omega^1_C)$ with $[\phi, \phi] = 0$. And the reduction to P of the pair (E, ϕ) is another pair $(E_P, \overline{\phi})$ such that $\phi_{|E_P} \simeq \overline{\phi} \in \Gamma(\mathrm{Ad}(E_P) \otimes \Omega^1_C)$.

In the sense of Definition 3.15, we prove here that for any semistable Higgs G-bundle on a connected smooth projective curve C there exists a Jordan–Hölder Higgs reduction such that its extension of the structure group to G is stable and unique up to isomorphism.

2. Notation and preliminaries

We recall here some definitions and well-known results for principal (Higgs) bundles (see [3] for more details). Let G be a connected reductive algebraic group over $\mathbb{C}$, Z the center of G, $Z_0 = \mathrm{Rad}(G)$ the radical of the group which is the connected component of Z containing e and let $G' = [G, G]$ be the derived subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the algebra of G' which is the derived algebra of G and $\mathfrak{z}$ the center algebra of Z in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ and $G = \mathrm{Rad}(G) \ltimes G'/D$, where D is a finite subgroup of G. Let T be a maximal torus on G' and $\mathfrak{t}$ its Lie algebra, we shall consider roots, weights and characters as forms on $\mathfrak{t}$ extended by zero on $\mathfrak{z}$, hence a dominant character of a parabolic subgroup is trivial in Z_0. We denote by

- $\Phi = \{\alpha_1, \dots, \alpha_l\}$ a simple system of roots of G;
- $\Delta(\Phi)$ a root system of G generated by Φ;
- $\Lambda(\Phi) = \{\lambda_1, \dots, \lambda_l\}$ the set of fundamental dominant weights of G associated to Φ;
- $D(\Phi)$ the set of weights of G associated to Φ.

Then:

- $\mathfrak{g}' = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\Phi)} \mathfrak{g}_\alpha$;
- for a parabolic subgroup P_I in G, where $I \subset \Phi$, we mean the group whose Lie algebra is $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(I)} \mathfrak{g}_\alpha$, where $\Delta(I) = \{\alpha \in \Delta \mid \alpha = \Sigma_{i=1}^{l} m_i \alpha_i \text{ with } m_j \geq 0 \text{ if } \alpha_j \notin I\}$. This can be written as $Z_0 P'$, where P' is the parabolic subgroup of $G' = [G, G]$ corresponding to $\mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(I)} \mathfrak{g}_\alpha$;
- the correspondence $P \mapsto P \cap G'$ and $P' \mapsto Z_0 P'$ gives a bijection between parabolic subgroups of G and G'.

Let C be a connected smooth projective curve and $\pi: E \to C$ a principal G-bundle on C. Let $\rho: G \to Gl(V)$ be a representation of G into the general linear group of a finite dimensional complex vector space V. Then

- the associated bundle $E(V) = E \times_\rho V$ is the quotient of $E \times V$ under the action of G given by $(u, v) \mapsto (ug, \rho(g^{-1})v)$ for $g \in G$;
- If $V = \mathfrak{g}$ is the Lie algebra of G, and ρ is the representation of the adjoint action of G on $\mathfrak{g}$, one gets the adjoint bundle of E, denoted by $\mathrm{Ad}(E)$;
- Let G_1 be another complex connected algebraic group; if ρ is a group homomorphism $\lambda: G \to G_1$, the associated bundle

$$E(\lambda) = E \times_\lambda G_1 = E_{G_1} \tag{1}$$

is a principal G_1-bundle. We say that the structure group G of E has been extended to G_1.

If E is a principal G-bundle on C, and F a principal G_1-bundle on C, a morphism $E \to F$ is a pair $\overline{f} = (\underline{f}, \lambda)$, where $\lambda: G \to G_1$ is a group homomorphism, and $f: E \to F$ is a morphism of bundles on C which is λ-equivariant, i.e., $\overline{f}(ug) = f(u)\lambda(g)$. This induces a vector bundle morphism $\tilde{f}: \mathrm{Ad}(E) \to \mathrm{Ad}(F)$ given by $\tilde{f}(u, \alpha) = (f(u), \lambda_*(\alpha))$, where $\lambda_*: \mathfrak{g} \to \mathfrak{g}_1$ is the morphism induced on the Lie algebras.

Let K be a closed subgroup of G; a *reduction* of the structure group G of E to K is a principal K-bundle F over C together with an injective K-equivariant bundle morphism $F \to E$. Using the natural action of G on the homogeneous space G/K, we denote by $E(G/K)$ the bundle over C with standard fibre G/K associated to E. Since there is an isomorphism $E(G/K) \simeq E/K$ of bundles over C, the reductions of the structure group of E to K are in a one-to-one correspondence with sections $\sigma: C \to E(G/K) \simeq E/K$.

We will denote by $E_\sigma = \sigma^*(E)$ the reduced principal K-bundle given by the section $\sigma: C \to E(G/K) \simeq E/K$.

Ramanathan's definition [4] of semistable principal G-bundle over a curve is the following:

Definition 2.1. Let E be a principal G-bundle on a smooth connected projective curve C and $T_{E/K,C}$ be the vertical tangent bundle to the bundle $\pi_K: E/K \to C$; E is said to be stable (resp. semistable) if for every proper parabolic subgroup $P \subset G$, and every reduction $\sigma: C \to E/P$, the pullback $\sigma^*(T_{E/P,C})$ has positive (resp. nonnegative) degree.

We switch now to principal Higgs G-bundles.

Definition 2.2. A principal Higgs G-bundle $\mathfrak{E}$ is a pair (E, ϕ), where E is a principal G-bundle, and ϕ is a global section of $\mathrm{Ad}(E) \otimes \Omega^1_C$ such that $[\phi, \phi] = 0$.