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Equivariant quantization of orbifolds

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Equivariant quantization is a new theory that highlights the role of symmetries in the relationship between classical and quantum dynamical systems. These symmetries are also one of the reasons for the recent interest in quantization of singular spaces, orbifolds, stratified spaces, etc. In this work, we prove the existence of an equivariant quantization for orbifolds. Our construction combines an appropriate desingularization of any Riemannian orbifold by a foliated smooth manifold, with the foliated equivariant quantization that we built in Poncin et al. (2009) [19]. Further, we suggest definitions of the common geometric objects on orbifolds, which capture the nature of these spaces and guarantee, together with the properties of the mentioned foliated resolution, the needed correspondences between singular objects of the orbifold and the respective foliated objects of its desingularization.

1. Introduction

Equivariant quantization, see [1–8], is the fruit of a recent research program that aimed at a complete and unambiguous geometric characterization of quantization. The procedure highlights the primary role of symmetries in the relationship between classical and quantum dynamical systems. One of the major achievements of equivariant quantization is the understanding that a fixed G-structure of the configuration space of a mechanical system guarantees existence and uniqueness of a G-equivariant quantization. Roughly and more generally, an equivariant, or better, a natural quantization of a smooth manifold M is a vector space isomorphism

$$\mathcal{Q}[\nabla] : \mathrm{Pol}(T^*M) \ni s \to \mathcal{Q}[\nabla](s) \in \mathcal{D}(M)$$

that maps a smooth function $s \in \mathrm{Pol}(T^*M)$ of the phase space T^*M, which is polynomial along the fibers, to a differential operator $\mathcal{Q}[\nabla](s) \in \mathcal{D}(M)$ that acts on functions $f \in C^\infty(M)$ of the configuration space M. The quantization map $\mathcal{Q}[\nabla]$ depends on the projective class $[\nabla]$ of an arbitrary torsionless connection ∇ of M, and it is equivariant with respect to the action of local diffeomorphisms ϕ of M, i.e.

$$\mathcal{Q}[\phi^*\nabla](\phi^*s)(\phi^*f) = \phi^*(\mathcal{Q}[\nabla](s)(f)),$$

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$\forall s \in \mathrm{Pol}(T^*M), \forall f \in C^\infty(M)$. Such natural and projectively invariant quantizations, or simply equivariant quantizations, were investigated in several works; see e.g. [9–11].

On the other hand, *quantization of singular spaces*, see e.g. [12–17], is an upcoming topic in Mathematical Physics, in particular in view of the interest of reduction for complex systems with symmetries. More precisely, if a symmetry group acts on the phase space or the configuration space of a general system, the quotient space is usually a singular space, an orbifold or a stratified space. The challenge consists in the quest for a quantization procedure of such singular spaces that in addition commutes with reduction.

It is now quite natural to ask which aspects of the new theory of equivariant quantization – that was recently extended from vector spaces to smooth manifolds – hold true for certain singular spaces. The main result of this work is the proof of existence of *equivariant quantization for orbifolds*.

A first difficulty of the attempt to construct an equivariant quantization on a singular space, is the proper definition of the actors in equivariant quantization – functions, differential operators, symbols, vector fields, differential forms, connections, etc. – for this space. Even in the case of orbifolds no universally accepted definitions can be found in literature. Moreover, geometric and algebraic definitions do not always coincide as in the classical context. Our method is based on the resolution of orbifolds proposed in [18]. More precisely, we combine this desingularization technique, which allows identifying any Riemannian orbifold V with the leaf space of a foliated smooth manifold $(\tilde{V}, \mathcal{F})$, with the foliated equivariant quantization that we constructed in [19], to build a singular equivariant quantization of orbifolds. To realize this idea, meaningful definitions, which not only capture the nature of orbifolds but ensure simultaneously that singular objects of V are in one-to-one correspondence with the respective foliated objects of $(\tilde{V}, \mathcal{F})$, are needed. We show that the chosen foliated resolution of orbifolds has exactly the properties that are necessary for this kind of relationship.

The paper is organized as follows. In the second section, we recall the definitions of foliated objects and of a foliated equivariant quantization. In the third, we detail our geometric definitions of singular objects on orbifolds and study their relevant properties for the singular equivariant quantization problem. We describe and further investigate, in Section 4, the foliated desingularization of a Riemannian orbifold, putting special emphasis on aspects that are of importance for the mentioned appropriate correspondence between foliated and singular objects. The last section deals with existence and the explicit construction of a singular equivariant quantization of Riemannian orbifolds.

2. Foliated quantization

In what follows, $(M, \mathcal{F})$ denotes an n-dimensional smooth manifold endowed with a regular foliation $\mathcal{F}$ of dimension p and codimension $q = n - p$. Moreover, U is an open set of $(M, \mathcal{F})$.

Let us first recall the definitions of the foliated objects and of the foliated natural and projectively invariant quantization given in [19]:

Definition 1. A *foliated function* f on U is a smooth function $f \in C^\infty(U)$ such that f is constant along the connected components of the traces of the leaves in U. In other words, if $(V, (x, y))$ is a system of adapted coordinates such that $V \cap U \neq \emptyset$, the local form of f on $U \cap V$ depends only on the transverse coordinates y.

We denote by $C^\infty(U, \mathcal{F})$ the algebra of all foliated functions of $(U, \mathcal{F})$.

Definition 2. A *foliated differential operator* D of order $k \in \mathbb{N}$ of U is an endomorphism of the space $C^\infty(U, \mathcal{F})$ of foliated functions, which reads in any system $(V, (x^1, \ldots, x^p, y^1, \ldots, y^q))$ of adapted coordinates in the following way:

$$D|_{U\cap V} = \sum_{|\alpha|\leq k} D_\alpha\, \partial_{y^1}^{\alpha^1} \ldots \partial_{y^q}^{\alpha^q},$$

where the coefficients $D_\alpha \in C^\infty(U \cap V, \mathcal{F})$ are locally defined foliated functions and where k is independent of the considered chart.

We denote by $\mathcal{D}^k(U, \mathcal{F})$ the $C^\infty(U, \mathcal{F})$-module of all k-th order foliated differential operators of $(U, \mathcal{F})$ and set

$$\mathcal{D}(U, \mathcal{F}) := \bigcup_{k\in\mathbb{N}} \mathcal{D}^k(U, \mathcal{F}).$$

The graded space $\mathcal{S}(U, \mathcal{F})$ associated with the filtered space $\mathcal{D}(U, \mathcal{F})$,

$$\mathcal{S}(U, \mathcal{F}) := \bigoplus_{k\in\mathbb{N}} \mathcal{S}^k(U, \mathcal{F}) := \bigoplus_{k\in\mathbb{N}} \mathcal{D}^k(U, \mathcal{F})/\mathcal{D}^{k-1}(U, \mathcal{F}),$$

is the space of *foliated symbols*. The k-th order symbol of a k-th order foliated differential operator D is then simply its class $\sigma_k(D)$ in the k-th term of the symbol space. The principal symbol $[D]$ of D is the symbol $\sigma_k(D)$ with the lowest possible k.

Definition 3. An *adapted vector field* of U is a vector field $X \in \mathrm{Vect}(U)$ such that $[X, Y] \in \Gamma(T\mathcal{F})$, for all $Y \in \Gamma(T\mathcal{F})$.