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Strange attractors in a chaotic coin flip simulation

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Abstract

Presented is a computer simulation used to model a variation of the game known as the gambler's ruin. A rich player gambles with a set amount of money m. The poor player starts out with zero capital, and is allowed to flip a coin in order to try to win the money. If the coin is heads, the poor player wins a dollar but if it is tails, the player loses a dollar. The poor player is always allowed to win the first flip, and is allowed to flip n times, even when the amount of money lost reaches zero. The dynamics of this process is chaotic due to fluctuations in the variance of the amount of money.

1. Introduction

Variations in the gambler's ruin, a random walk problem, have been extensively studied. For example, Rocha and Stern [1] consider an asymmetric n player problem where everyone starts out with equal initial fortunes. Kmet and Petkovsek [2] examine a multi-dimensional version, where each of the players have two currencies apiece. Itoh et al. [3] study a variation with three players, where the game continues until the banker is ruined or until one of the three wins all of the money. The gambler's ruin problem has found applications in genetics [4], solar physics [5], solid state physics [6], and quantum mechanics [7,8].

The chaotic behavior of a coin toss apart from the gambler's ruin has also resulted in numerous studies, from classical to quantum chaos [9–12]. The fractal nature of the classical gambler's ruin problem, which is Brownian motion, is well known; for example it is analyzed by Schroeder [13].

Coin toss problems involving the exchange of money have also been of interest to economists, due to changes in the standard deviation of the sums involved. In finance, fluctuating standard deviations are known as volatility. Volatility, fractals, and chaos in the financial markets have been of interest for decades [14–16]. One example is shown by Liu et al., who study the statistical properties of the volatility of price fluctuations in the S&P 500 index of the New York Stock Exchange, by using a time series analysis to examine power spectrums, probability distributions, and autocorrelation functions.

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The variance is related to the volatility because it is equal to the square of this standard deviation. Fractal time random walks which are not pure Brownian motion, due to the behavior of moments such as variances, are noted by Shlesinger et al. [17,18], Santini [19], and Scafetta and Grigolini [20].

The current research presents a variation on the gambler's ruin, whereby the conditions imposed on the problem result in chaotic behavior instead of pure Brownian noise. This variation was developed by Yorke, who has been known to extend the theory of chaos to the creation of games [21]. As in Levy flights, the chaotic behavior is due to the variance, which fluctuates with the number of coin tosses.

2. *Y* variation of the Gambler's ruin

The *Y* variation on the gambler's ruin is as thus: there are two players, one representing a rich man and the other a poor man. The rich man has a set amount of money *m*, that he wishes to gamble. The poor man is allowed to try to win this money by flipping an unweighted coin *n* amount of times, and the game ends only when *n* does. This means that even if the poor man at some point loses all of his money, he is allowed to continue flipping until *n* is reached. The poor man wins a dollar if the coin is heads, and loses a dollar if the coin is tails.

The generating equation that may be used to represent the game is

$$X_{n+1} = X_n + g_n \tag{1}$$

with $g_n = 1, -1, 0$. Eq. (1) represents the velocity, or incremental movement of the money. This motion, unlike the classical gambler's ruin, is confined to the three aforementioned states.

This variation differs from the typical random walk in the following ways:

(1) These states are not independent of each other; once zero is reached, the state afterwards is directly dependent upon it, thus giving the system at certain stages memory.
(2) Because of the *Y* conditions, certain transitions are forbidden.

We can represent the possible states by + (the poor man gains a dollar), − (the poor man loses a dollar), or 0 (the poor man loses all of his money, so the flow of money is stopped). The *Y* variation then always forbids sequences where a zero is followed by a minus—that is to say, sequences of the type 0 −. Table 1 shows a set of examples for forbidden sequences for n = three flips. Sequence 1 shows three flips of 1, 1, 0, representing win, win, and then a loss of all of the money, which is physically impossible. The remaining sequences, 2–5 all have states 0 −, which is never allowed.

The forbidden sequences clearly show the system has a type of memory, since future states have probabilities dependent as far back as two previous flips. This represents what is known as a semi-Markovian process. This is different from the classical random walk, which is a pure Markovian process whereby future states have no bearing on past ones.

The coin flips were generated using a computer program which was originally written in Matlab and then ported to Perl. The program simulates the gambler's ruin from the side of the poor man, who is additionally allowed to always win the first toss. The amount of money, *X*, the rich man gambles is set to $1k (one thousand dollars). The number of flips, *n*, varies from 100, 10^2, 10^3, 10^4, to 10^5 (one million). Ten games are run per *n*, for a total of 50 games in all, and where appropriate, the average of the games is used to compute statistics. Fig. 1 shows a typical game, where the amount of money won by the poor man is shown on the *y*-axis, and the number of flips is shown on the *x*-axis. In order to show detail, *n* is limited to 1000 flips, but the pattern for all *n* is the same, though the scales are different. For all of the games, the average time for one flip as calculated on a computer with a Pentium processor is .0014 s.

Table 1
Forbidden transitions in the Yorke variation of the Gambler's ruin three state examples

Sequence	State 1	State 2	State 3
1	+	+	0
2	0	–	–
3	0	–	0
4	0	–	+
5	0	0	–