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Fibonacci matrices, a generalization of the "Cassini formula", and a new coding theory

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Abstract

We consider a new class of square Fibonacci $(p+1)\times(p+1)$-matrices, which are based on the Fibonacci p-numbers $(p = 0, 1, 2, 3, \ldots)$, with a determinant equal to $+1$ or -1. This unique property leads to a generalization of the "Cassini formula" for Fibonacci numbers. An original Fibonacci coding/decoding method follows from the Fibonacci matrices. In contrast to classical redundant codes a basic peculiarity of the method is that it allows to correct matrix elements that can be theoretically unlimited integers. For the simplest case the correct ability of the method is equal 93.33% which exceeds essentially all well-known correcting codes.

1. Introduction

In the last decades the theory of Fibonacci numbers [1,8] was complemented by the theory of the so-called *Fibonacci Q-matrix* [1,2]. The latter is a square 2×2 matrix of the following form:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

In [1] the following property of the nth power of the Q-matrix was proved

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \tag{2}$$

$$\operatorname{Det} Q^n = F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \tag{3}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots, F_{n-1}, F_n, F_{n+1}$ are Fibonacci numbers given with the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1} \tag{4}$$

with the initial terms

$$F_1 = F_2 = 1. \tag{5}$$

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Note that identity (4) is called "*Cassini formula*" in honor of the well-known 17th century astronomer Giovanni Cassini (1625–1712) who derived this formula.

In 1977 the author introduced so-called *Fibonacci p-numbers* [3]. For a given integer $p = 0, 1, 2, 3, \ldots$ the Fibonacci p-numbers are given with the following recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad \text{with } n > p+1 \tag{6}$$

with the initial terms

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1. \tag{7}$$

In [4] the notion of the Q_p-matrices ($p = 0, 1, 2, 3, \ldots$) was introduced. This notion is a generalization of the Q-matrix (1) and is connected to the Fibonacci p-numbers (6) and (7).

The main purpose of the present article is to develop a theory of the Q_p-matrices. The next purpose is to give a generalization of the "Cassini formula" (4) that follows from the theory of the Q_p-matrices. Also a new approach to a coding theory, which is based on the Q_p-matrices, is considered.

2. Some properties of the Fibonacci p-numbers

It is clear that the recurrence formula (6) with the initial terms (7) "generates" an infinite number of recurrent sequences. In particular, for the case $p = 0$ recurrence relation (6) and (7) reduces to the following:

$$F_0(n) = F_0(n-1) + F_0(n-1) \quad \text{with } n > 1, \tag{8}$$

$$F_0(1) = 1. \tag{9}$$

This recurrence relation "generates" the binary numbers: $1, 2, 4, 8, \ldots, 2^{n-1} \ldots$.

For the case $p = 1$ recurrence relation (6) and (7) reduces to the following:

$$F_1(n) = F_1(n-1) + F_1(n-2) \quad \text{with } n > 2, \tag{10}$$

$$F_1(1) = F_1(2) = 1. \tag{11}$$

This recurrent relation "generates" the classical Fibonacci numbers $F_1(n) = F_n$

$$1, 1, 2, 3, 5, 8, 13, \ldots \tag{12}$$

It is clear that for general case the recurrence relation (6) and (7) "generates" infinite number of numerical series, which are a wide generalization of the classical Fibonacci numbers.

Like to the classical Fibonacci numbers (12) the Fibonacci p-numbers for the case $p > 0$ allow their extension to the negative values of the argument n. For calculation of the Fibonacci p-numbers $F_p(0), F_p(-1), F_p(-2), \ldots, F_p(-p), \ldots, F_p(-2p+1)$ we will use recurrence relation (6) and initial terms (7). Representing the Fibonacci p-numbers $F_p(p+1)$ in the form (6) we get

$$F_p(p+1) = F_p(p) + F_p(0). \tag{13}$$

As according to (7) $F_p(p) = F_p(p+1) = 1$ it follows from (13) that $F_p(0) = 0$.

Continuing this process, that is, representing the Fibonacci p-numbers $F_p(p), F_p(p-1), \ldots, F_p(2)$ in the form (6) we get

$$F_p(0) = F_p(-1) = F_p(-2) = \cdots = F_p(-p+1) = 0. \tag{14}$$

Let us represent now the number $F_p(1)$ in the form:

$$F_p(1) = F_p(0) + F_p(-p). \tag{15}$$

As $F_p(1) = 1$ and $F_p(0) = 0$ it follows from (15) that

$$F_p(-p) = 1. \tag{16}$$

Representing the Fibonacci p-numbers $F_p(0), F_p(-1), \ldots, F_p(-p+1)$ in the form (6) we can find

$$F_p(-p-1) = F_p(-p-2) = \cdots = F_p(-2p+1) = 0. \tag{17}$$

Continuing this process we can get all values of the Fibonacci p-numbers $F_p(n)$ for the negative values of n (see Table 1).