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Diffusion as a result of transition in behavior of deterministic maps

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Abstract

A transition from nondiffusive growth of variance of a particles position to diffusive one of various type, in nonlinear deterministic maps, is discussed. The details of generating regular, subdiffusive and superdiffusive cases are shown.

1. Introduction

Deterministic diffusion is an interesting phenomenon of todays nonlinear science. Under its notion we understand a deterministic model of a system, generating behavior that is statistically equivalent to one, described by transport equations like Fokker–Planck, or Smoluchowski equation, i.e. is capable of generating proper scaling of variance with time, and introduce drift [1]. The origin of diffusion is switched in these models from stochastics into deterministic nonlinear behavior on microscale [2,3].

In spite of quite a few papers, devoted to the problem [4–11], some aspects of deterministic diffusion are still not clear enough. One of such aspects is the detailed analysis of the origins of diffusive motion in nonlinear maps, organized into periodic cells (i.e. where a map, defined on region $[0,1]$ is extended, by repeating itself modulo 1, into the whole real axis).

As an example, illustrating the problem, consider a chaotic map. In such a map the nearby trajectories must separate in exponential manner, according to Lyapunov exponent [4,12–14]. The problem we would like to deal with in this paper is how to arrive at diffusion (which is governed by nonexponential growth of displacement between trajectories) using such microscopic behavior, in case of both chaotic and nonchaotic dynamical systems. For doing this, we provide an approach that is different than classical methods, developed for diffusion in chaotic systems, like escape rate formalisms, spectral analysis [7,15–18], Green–Kubo equations [8,9], KAM theorems [10,11], etc. The mentioned approaches are very fruitful tools for chaotic dynamics analysis, but they operate on an abstract level that is too high to see all the details.

An example of the transition from nonlinear regime to a diffusive growth of variance is shown in Fig. 1. The map shown here is defined by the equation

$$\Delta x_n = 2\sin 2\pi x_n \quad (1)$$

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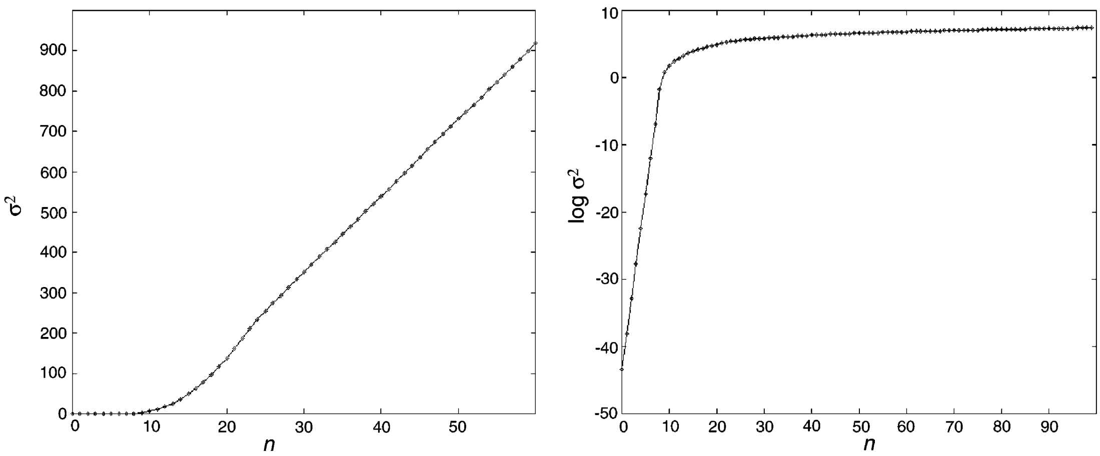

Fig. 1. The variance growth with respect to time for a sine map. Presented in logarithmic and normal coordinates.

We can see an initial growth of exponential type (the initial variance is of the order 10^{-11}, followed by linear growth. Similar transition can be observed in nonlinear systems that are not chaotic, as in the paper [3]. The transition takes place at about ninth iteration, which corresponds to a standard deviation of about one. This indicates a connection between regime change and the escape from the period $[0,1]$.

2. The properties of a diffusive map

To describe and model diffusion, using dynamical system, we need to have a mapping that generates some new positions and is discrete in time, to resemble the random molecular motion i.e. the Brownian dynamics according to which a zig-zag trajectory is generated.

Well-known example of such a map is the logistic map [19],

$$x_{n+1} = rx_n(1 - x_n) \tag{2}$$

In this map, r stands for a control parameter that describes, whether the equation is in chaotic regime, or not. In chaotic regime for two neighboring initial conditions this map is able to generate exponentially divergent trajectories. The necessary and sufficient condition for that is to have the slope of map at some point x greater than one (Fig. 2). Only in such case it is possible to expand the ϵ difference between initial conditions, i.e.

$$\epsilon_{n+1} = s\epsilon_n \tag{3}$$

where s stands for the slope of map. However, this separation does not have to always obey the Lyapunov exponential divergence as was shown in [3] for the case with separation governed by the Heaviside like step function. It should also be noted that the exponential divergence is considered in the average, since there may appear steps, where the trajectories do not separate (e.g. on the flat part of parabola in logistic map).

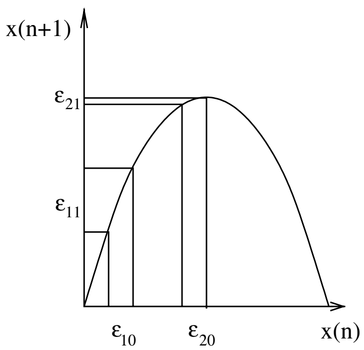

Fig. 2. Illustration on how does the slope influence the trajectory divergence. ϵ_{10} gets expanded, and ϵ_{20} gets contracted.