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A hierarchy of nonlinear lattice soliton equations, its integrable coupling systems and infinitely many conservation laws

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Abstract

A hierarchy of nonlinear integrable lattice soliton equations is derived from a discrete spectral problem. The lattice hierarchy is proved to have discrete zero curvature representation. Moreover, it is shown that the hierarchy is completely integrable in the Liouville sense. Further, we construct integrable couplings of the resulting hierarchy through an enlarging algebra system $\widetilde{X}$. At last, infinitely many conservation laws of the hierarchy are presented.

1. Introduction

In recent years, the study of integrable lattice equation has aroused increasing interest [1–12]. Many nonlinear integrable lattice equations have been proposed, such as the Ablowitz–Lodik lattice, the Toda lattice, the discrete KdV equation, and so on. Such integrable lattice equations play an important role in a number of contexts and have extensive applications in mathematical physics, biology, optics, discrete geometry and cellular automata. In soliton theory, an important subject is to search for new Lax integrable lattice hierarchies. Moreover, the algebraic structure is also very important in investigating nonlinear integrable lattice systems. Such as the discrete zero curvature representation [10], the Hamiltonian structure [11], infinitely many conservation laws [13–15], and so on. In addition, integrable coupling is a new and significant direction in soliton theory, which originated from investigation of centerless Virasoro symmetry algebras [21], and can be used for obtaining some new integrable hierarchies [18–21].

It is well-known that the method of Lax pair is an important way to generate new lattice hierarchies [17]. However, since there are no commutative operations in the discrete zero curvature equation, which is different from the continuous case, we cannot directly employ the Loop algebras to construct discrete equations. And we cannot extend the

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theory on the continuous integrable coupling to the discrete case. Taking account of the shortcoming, we try to construct a novel algebraic system X and its extended system $\widetilde{X}$ to solve the problem. New algebraic systems are based on the known Loop algebra and the properties of difference operators [16,17].

This paper is organized as follows. In Section 2, we derive a hierarchy of nonlinear integrable discrete soliton equation from a discrete spectral problem. It is shown that the hierarchy is completely integrable in the Liouville sense. In Section 3, by constructing a new Loop algebra system $\widetilde{X}$, integrable couplings of the resulting hierarchy are constructed. In Section 4, we establish infinitely many conservation laws for the resulting hierarchy. Finally, conclusions and remarks are given.

2. A new algebraic system and a hierarchy of discrete soliton equations

First, we specify some fundamental conceptions. The shift operator E, the inverse of E and two difference operator D and Δ are defined as follows:

$$(Ef)(n) = f(n+1), \qquad (E^{-1}f)(n) = f(n-1), \quad n \in Z, \tag{1a}$$

$$(Df)(n) = f(n+1) - f(n), \quad (\Delta f)(n) = f(n+1) - f(n-1) n \in Z, \tag{1b}$$

where f is a lattice function, i.e., a function from Z to R. As normal, we write $f^{(j)} = E^j f, j \in Z$. We assume that $u = (r, s)^{\mathrm{T}}$, $r = r(n, t)$, $s = s(n, t)$ are real functions defined over $Z \times R$, and u is required to vanish rapidly at the infinity. λ is the spectral parameter and $\lambda_t = 0$.

A lattice equation

$$u_t = K(u, Eu, E^{-1}u, \ldots) \tag{2}$$

is said to be Lax integrable, if it can be rewritten as a compatibility condition

$$U_t = (EV)U - UV \tag{3}$$

of a discrete spatial spectral problem

$$E\phi = U(u, \lambda)\phi \tag{4}$$

and a corresponding continuous time evolution equations

$$\phi_t = V(u, \lambda)\phi, \tag{5}$$

where $U(u, \lambda)$ and $V(u, \lambda)$ are of same order square matrices. Eqs. (4) and (5) are said to be a Lax pair of Eq. (2). Eq. (3) is called a discrete zero curvature representation of Eq. (2).

The Gateaux derivative, the variational derivative and the inner product are defined by

$$J'(u)[v] = \frac{\partial}{\partial \varepsilon} J(u + \varepsilon v)|_{\varepsilon=0},$$

$$\frac{\delta \widetilde{H}}{\delta u} = \sum_{m \in Z} E^{-m}\left(\frac{\partial H}{\partial u^{(m)}}\right),$$

$$\langle f, g\rangle = \sum_{n \in Z} (f(n), g(n)),$$

where $f = f(n)$ and $g = g(n)$ are required to be rapidly vanishing at the infinity, $(f(n), g(n))$ denotes the standard inner product $f = f(n)$ and $g = g(n)$ in the Euclidean space R^2. $\widetilde{H} = \sum_{n \in Z} H(u(n))$. Operator J^* is defined by $\langle f, J^* g\rangle = \langle Jf, g\rangle$, which is called the adjoint operator of J.

If an operator J has the property $J = -J^*$, then J is called to be skew-symmetric. A linear operator J is called a Hamiltonian operator, if J is a skew-symmetric operator satisfying the Jacobi identity, i.e.,

$$\langle J'(u)[Jf]g, h\rangle + \mathrm{Cycle}(f, g, h) = 0.$$

The associated Poisson bracket with a given Hamiltonian operator J is given by

$$\{f, g\}_J = \left\langle \frac{\delta f}{\delta u}, J \frac{\delta g}{\delta u} \right\rangle. \tag{6}$$