Journal of Geometry and Physics 60 (2010) 513–530

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

The k-Cauchy–Fueter complex, Penrose transformation and Hartogs phenomenon for quaternionic k-regular functions☆

Wei Wang

Department of Mathematics, Zhejiang University, Zhejiang 310027, PR China

ARTICLE INFO

Article history:
Received 10 March 2009
Received in revised form 5 November 2009
Accepted 29 November 2009
Available online 3 December 2009

MSC:
30G35
35N05
32L25
32L10
58J10

Keywords:
k-Cauchy–Fueter complex
Penrose transformation
Quaternionic k-regular functions
Elliptic differential complex
Non-homogeneous k-Cauchy–Fueter equations
Hartogs phenomenon
Integral representation formula

ABSTRACT

By using complex geometric method associated to the Penrose transformation, we give a complete derivation of an exact sequence over $\mathbb{C}^{4n}$, whose associated differential complex over $\mathbb{H}^n$ is the k-Cauchy–Fueter complex with the first operator $D_0^{(k)}$ annihilating k-regular functions. $D_0^{(1)}$ is the usual Cauchy–Fueter operator and 1-regular functions are quaternionic regular functions. We also show that the k-Cauchy–Fueter complex is elliptic. By using the fundamental solutions to the Laplacian operators of 4-order associated to the k-Cauchy–Fueter complex, we can establish the corresponding Bochner–Martinelli integral representation formula, solve the non-homogeneous k-Cauchy–Fueter equations and prove the Hartogs extension phenomenon for k-regular functions in any bounded domain.

1. Introduction

1.1. The k-Cauchy–Fueter operator

In [1], we show the Cauchy–Fueter complex to be elliptic, solve the non-homogeneous Cauchy–Fueter equations, prove the Hartogs extension phenomenon for quaternionic regular functions, and derive the quaternionic version of Bochner–Martinelli integral representation formula (but see also [2–6]). The Cauchy–Fueter operator is the second one of a family of k-Cauchy–Fueter operators $D_0^{(k)}$, $k = 0, 1, \ldots$. The purpose of this paper is to derive the corresponding k-Cauchy–Fueter complex completely and to establish the above results for general k.

The *affine Minkowski space* can be embedded in $\mathbb{C}^{2\times 2}$ by

$$(x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 + x_1 & x_2 + \mathrm{i}x_3 \\ x_2 - \mathrm{i}x_3 & x_0 - x_1 \end{pmatrix}, \tag{1.1.1}$$

$\mathrm{i} = \sqrt{-1}$, while the quaternionic algebra $\mathbb{H}$ can be embedded in $\mathbb{C}^{2\times 2}$ by

☆ Supported by National Nature Science Foundation in China (No. 10871172).
E-mail address: wwang@zju.edu.cn.

0393-0440/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2009.11.011

$$x_0+x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}\mapsto\begin{pmatrix}x_0+\mathrm{i}x_1 & -x_2-\mathrm{i}x_3\\ x_2-\mathrm{i}x_3 & x_0-\mathrm{i}x_1\end{pmatrix}. \tag{1.1.2}$$

They are different embeddings of $\mathbb{R}^4$ in $\mathbb{C}^4$. We will use the conjugate embedding

$$\iota:\mathbb{H}^n\simeq\mathbb{R}^{4n}\hookrightarrow\mathbb{C}^{2n\times 2},\qquad (q_0,\ldots,q_{n-1})\mapsto(z^{AA'}), \tag{1.1.3}$$

$A=0,1,\ldots,2n-1$, $A'=0,1$, with

$$\begin{pmatrix}z^{(2l)0'} & z^{(2l)1'}\\ z^{(2l+1)0'} & z^{(2l+1)1'}\end{pmatrix}:=\begin{pmatrix}x_{4l}-\mathrm{i}x_{4l+1} & -x_{4l+2}+\mathrm{i}x_{4l+3}\\ x_{4l+2}+\mathrm{i}x_{4l+3} & x_{4l}+\mathrm{i}x_{4l+1}\end{pmatrix}, \tag{1.1.4}$$

$l=0,\ldots,n-1$. Denote

$$\nabla_{AA'}:=\frac{\partial}{\partial z^{AA'}}, \tag{1.1.5}$$

the holomorphic derivatives on $\mathbb{C}^{4n}$. An element of $\mathbb{C}^2$ is denoted by $(\phi_{A'})$ with $A'=0,1$, while an element of the symmetric power $\odot^j\mathbb{C}^2$ is denoted by $(\phi_{A'B'\cdots C'})$ with $A',B',\ldots,C'=0,1$, where $\phi_{A'B'\cdots C'}$ is the same as that of the permutation of subscripts. An element of the exterior power $\Lambda^j\mathbb{C}^{2n}$ is denoted by $(\phi_{AB\cdots C})$ with $A,B,\ldots,C=0,1,\ldots,2n-1$, where $\phi_{AB\cdots C}$ is that of the permutation of subscripts multiplying the sign of the permutation. We also denote by ϕ_* functions valued in such vector spaces. Let

$$\begin{aligned}&D_0^{(k)}:C^\infty(\mathbb{C}^{4n},\odot^k\mathbb{C}^2)\longrightarrow C^\infty(\mathbb{C}^{4n},\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^{2n}),\\ &\phi_{A'B'\cdots C'}\longmapsto(D_0^{(k)}\phi)_{AB'\cdots C'}:=\nabla_A^{A'}\phi_{A'B'\cdots C'}.\end{aligned} \tag{1.1.6}$$

Here and in the following we use Einstein convention of taking summation over repeated indices. The repeated indices A' and A are taken over $0,1$ and $0,1,\ldots,2n-1$, respectively. The matrix

$$\epsilon=(\epsilon_{A'B'})=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{1.1.7}$$

is used to raise or lower indices, e.g. $\nabla_A^{A'}\epsilon_{A'B'}=\nabla_{AB'}$.

When $n=1$, pulling back to the affine Minkowski space by the embedding (1.1.1), $D_0^{(k)}\phi=0$ is the *helicity* $\frac{k}{2}$ *massless field* equations [7,8]. $D_0^{(1)}\phi=0$ is the Dirac–Weyl "equation of an electron" for mass zero whose solutions correspond to neutrinos, $D_0^{(2)}\phi=0$ is the Maxwell equation whose solutions correspond to photons, $D_0^{(3)}\phi=0$ is linearized Einstein's equation whose solutions correspond to "weak gravitational field", and so on.

Pulling back to the quaternionic space $\mathbb{H}^n\cong\mathbb{R}^{4n}$ by the embedding (1.1.4), we set

$$\begin{pmatrix}\widetilde{\nabla}_{(2l)0'} & \widetilde{\nabla}_{(2l)1'}\\ \widetilde{\nabla}_{(2l+1)0'} & \widetilde{\nabla}_{(2l+1)1'}\end{pmatrix}:=\begin{pmatrix}\partial_{x_{4l}}+\mathrm{i}\partial_{x_{4l+1}} & -\partial_{x_{4l+2}}-\mathrm{i}\partial_{x_{4l+3}}\\ \partial_{x_{4l+2}}-\mathrm{i}\partial_{x_{4l+3}} & \partial_{x_{4l}}-\mathrm{i}\partial_{x_{4l+1}}\end{pmatrix}, \tag{1.1.8}$$

on $\mathbb{R}^{4n}$. By abuse of notations, $\widetilde{\nabla}_{AA'}$ is also denoted by $\nabla_{AA'}$. We call $D_0^{(k)}:C^\infty(\mathbb{R}^{4n},\odot^k\mathbb{C}^2)\longrightarrow C^\infty(\mathbb{R}^{4n},\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^{2n})$, given by (1.1.6) with $\nabla_{AA'}$ provided by (1.1.8), the *k-Cauchy–Fueter operator*. The usual Cauchy–Fueter operator is

$$\overline{\partial}_{q_l}=\partial_{x_{4l}}+\mathbf{i}\partial_{x_{4l+1}}+\mathbf{j}\partial_{x_{4l+2}}+\mathbf{k}\partial_{x_{4l+3}},\quad l=0,\ldots,n-1.$$

For $f=f_0+f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}$, set $\phi^{0'}=f_0+\mathbf{i}f_1$, $\phi^{1'}=f_2-\mathbf{i}f_3$. It is known that the Cauchy–Fueter equation $\frac{\partial}{\partial\overline{q_l}}f=0$ can be written as

$$\begin{pmatrix}\nabla_{00'} & \nabla_{01'}\\ \nabla_{10'} & \nabla_{11'}\end{pmatrix}\begin{pmatrix}\phi^{0'}\\ \phi^{1'}\end{pmatrix}=0.$$

Briefly, it can be write as $\nabla_{AA'}\phi^{A'}=0$, which is equivalent to the 1-Cauchy–Fueter operator. For a domain Ω in $\mathbb{H}^n$, a function $f:\Omega\to\odot^k\mathbb{C}^2$ is called *left k-regular in Ω* (or briefly *k-regular in Ω*) if it satisfies

$$D_0^{(k)}f(q)=0 \tag{1.1.9}$$

for any $q\in\Omega$. The set of all k-regular functions in Ω is denoted by $\mathcal{H}_{(k)}(\Omega)$.

1.2. Construction of exact sequences over $\mathbb{C}^{4n}$ by complex geometric method

To obtain the k-Cauchy–Fueter complex, let us consider the *flag manifolds*:

$$\mathbb{F}_{d_1,\ldots,d_r}:=\{\,(L_1,\ldots,L_r);L_1\subset\cdots\subset L_r\text{ are linear subspaces of }\mathbb{C}^{2(n+2)}\text{ with }\dim_{\mathbb{C}}L_j=d_j\},$$