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On sufficient and necessary conditions for linearity of the transverse Poisson structure

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ABSTRACT

We study the possibility of bringing the transverse Poisson structure to a coadjoint orbit (on the dual of a real Lie algebra) to a normal linear form. We study the relation between two sufficient conditions for linearity of such structures (P. Molino's condition and our own). We then use these conditions to conclude that, if the isotropy subgroup of the (singular) point in question is compact, or if the isotropy subalgebra is semisimple, then there is a linear transverse Poisson structure to the corresponding coadjoint orbit.

Finally, by using a natural necessary condition for linearity of such structures, we will prove that there is no polynomial transverse Poisson structure in the case of $\mathfrak{e}(3)^*$.

1. Introduction

A real Poisson manifold is a pair $(M, \{,\})$, where M is a real, finite-dimensional, smooth manifold and $\{,\}$ is a Lie algebra structure on $C^\infty(M)$ satisfying the Leibniz identity:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \forall f, g, h \in C^\infty(M).$$

The simplest examples are: (a) symplectic manifolds with their induced Poisson bracket and (b) the dual of any real Lie algebra with the Lie–Poisson structure.

The notion of *symplectic leaf through a point of a Poisson manifold* was introduced by Weinstein in [1], together with the notion of *transverse Poisson structure to the symplectic leaf.*

In the particular case where the starting Poisson manifold is the dual of any Lie algebra $\mathfrak{g}$ with its Lie–Poisson structure, the symplectic leaf through a point $\mu \in \mathfrak{g}^*$ is just the coadjoint orbit of μ. In such a case the following can be taken as a transverse manifold to the coadjoint orbit [1,2]:

$$N = \mu + \mathfrak{h}^\circ,$$

where $\mathfrak{h}$ is any supplement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$ ($\mathfrak{g}_\mu$ denotes the isotropy subalgebra of μ) and $\mathfrak{h}^\circ$ stands for the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$.

On such a transverse manifold there is a canonically defined Poisson structure [1] which is precisely the so-called *transverse Poisson structure to the coadjoint orbit of* μ. Of course, due to the choice involved (choice of the supplement $\mathfrak{h}$ on which N depends), one can obtain different transverse Poisson structures to the coadjoint orbit of μ. Nevertheless, two

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transverse Poisson structures to the coadjoint orbit of μ will always be Poisson-diffeomorphic [1]. And this brings us to the following question: how can we choose the supplement $\mathfrak{h}$ so that the transverse Poisson structure to the coadjoint orbit of μ is "as simple as possible"? For example, can we choose $\mathfrak{h}$ so that such structure is linear?

Remark 1. In a strict sense, *linear Poisson structures* are defined only on vector spaces, which is not the case for $N = \mu + \mathfrak{h}^\circ$ (unless μ equals zero). Nevertheless, on an affine space such as $N = \mu + \mathfrak{h}^\circ$, it is common to define as *linear*, any Poisson structure whose expression is linear on a system of affine coordinates on N.

As usual, one says that a Poisson structure on N is *linearizable at* μ, if it is locally (around μ) Poisson-diffeomorphic to a linear Poisson structure.

As a consequence of the Poisson-equivalence of all transverse Poisson structures to the same orbit, if there is a linear transverse Poisson structure on some $N = \mu + \mathfrak{h}^\circ$, then any other transverse Poisson structure (on any other N) will be Poisson-diffeomorphic to the linear one, so it will be *linearizable at* μ. This can be used as a "necessary condition" for the existence of a linear transverse Poisson structure.

We can, likewise, define a *polynomial* Poisson structure on $N = \mu + \mathfrak{h}^\circ$, to be any Poisson structure on N whose expression is polynomial on a system of affine coordinates on N. Naturally, a Poisson structure on N is said to be *polynomializable at* μ, if it is locally Poisson-diffeomorphic to a polynomial Poisson structure.

Around 1984 Molino showed (see [3]) that, if the supplement $\mathfrak{h}$ satisfies:

$$[\mathfrak{g}_\mu, \mathfrak{h}] \subset \mathfrak{h} \tag{1}$$

then the transverse Poisson structure on $N = \mu + \mathfrak{h}^\circ$ is linear. Using Molino's condition, we gave, in [4], a sufficient condition (on the isotropy subalgebra $\mathfrak{g}_\mu$) for the existence of a linear transverse Poisson structure.

In Section 2 we will "relax" the condition we produced in [4], and study its relation with Molino's sufficient condition (1).

We proceed to give an example showing that neither of these two sufficient conditions is necessary.

We will then prove that, if: (a) $\mathfrak{g}$ is of compact type or (b) $\mathfrak{g}_\mu$ is semisimple or (c) G_μ (the isotropy subgroup of μ) is compact, then the transverse Poisson structure on a convenient $N = \mu + \mathfrak{h}^\circ$ will be linear.

Finally, in Section 3 we will clarify the situation of the transverse Poisson structure to any of the singular coadjoint orbits of $\mathfrak{e}(3)^*$. We will show that there is no *polynomial* transverse Poisson structure in such case.

2. On sufficient conditions for linearity of the transverse Poisson structure

We start by fixing some notation regarding the transverse Poisson structure to a coadjoint orbit. We refer the reader to [1,2,4] for a more detailed exposition.

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra and consider the Lie–Poisson structure on its dual space: $(M, P) = (\mathfrak{g}^*, L)$. Given $\mu \in \mathfrak{g}^*$ let O_μ denote the symplectic leaf through μ (this is just the coadjoint orbit of μ), and $\mathfrak{g}_\mu$ denote the isotropy subalgebra of μ:

$$\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} : \mu \circ ad_\xi = 0\}.$$

Pick any vector subspace $\mathfrak{h}$ such that, as vector spaces:

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h},$$

(we will refer to such $\mathfrak{h}$ as a *supplement of* $\mathfrak{g}_\mu$), and consider the following transverse manifold to O_μ:

$$N_\mathfrak{h} = \mu + \mathfrak{h}^\circ$$

($\mathfrak{h}^\circ$ stands for the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$). On such a manifold build the transverse Poisson structure to O_μ at μ and denote it by $(N_\mathfrak{h}, P_\mathfrak{h})$. Recall that this Poisson structure depends on the choice of $\mathfrak{h}$, even though different choices of $\mathfrak{h}$ will produce (locally) Poisson-equivalent structures.

We are interested in finding $\mathfrak{h}$ such that the transverse Poisson structure $(N_\mathfrak{h}, P_\mathfrak{h})$ is as simple as possible. More precisely, we want to consider the problem of finding $\mathfrak{h}$ so that $(N_\mathfrak{h}, P_\mathfrak{h})$ is linear, that is to say, $P_\mathfrak{h}$ is linear on a system of affine coordinates on $N_\mathfrak{h}$.

We start by proving a slightly stronger version of our theorem 3 from [4]. This will be done by relaxing the condition imposed on the bilinear form B in that theorem.

Definition 1. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_0$ be any subalgebra of $\mathfrak{g}$. We will say that a symmetric bilinear form B on $\mathfrak{g}$ is *$ad_{\mathfrak{g}_0}$-invariant* if:

$$B([\xi, \eta], \zeta) + B(\eta, [\xi, \zeta]) = 0, \quad \forall \xi \in \mathfrak{g}_0, \forall \eta, \zeta \in \mathfrak{g}.$$

Clearly *ad*-invariant bilinear forms on $\mathfrak{g}$ are also $ad_{\mathfrak{g}_0}$-invariant, for any Lie subalgebra $\mathfrak{g}_0$.

It is easy to see that the following results of [4] still hold with the "relaxed invariance of B":