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Review

Is the Hamiltonian geometrical criterion for chaos always reliable?

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ABSTRACT

It is found that the application of a newly developed geometrical criterion, in which negative eigenvalues of the associated matrix determined by the *dynamical curvature* of a conformal metric for a Hamiltonian system are used to identify the onset of local instability or chaos, is somewhat problematic in some circumstances. In fact, this criterion is neither necessary nor sufficient for the prediction of instability of orbits on a same energy hypersurface because it is not in good agreement with information on unstable or chaotic behavior given by the maximal Lyapunov exponent in general.

Contents

1. Introduction

In the last decade or so, much effort has been devoted to formulate invariant chaos indicators in general relativity (for reviews and references see [1–6]). For example, Sota et al. gave a geometrical criterion for chaos based on the eigenvalues associated with the Weyl curvature tensor [5]. The geometrical criteria like this are also applied to Hamiltonian systems in classical mechanics [7,8]. For isotropic manifolds given by a Hamiltonian system

$$H = \frac{1}{2m}\boldsymbol{p}^2 + V(\boldsymbol{x}) \tag{1}$$

with V as a potential function of space variables, the geodesic deviation equation takes a simple form

$$\frac{D^2\boldsymbol{J}}{ds^2} + K\boldsymbol{J} = 0, \tag{2}$$

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where D denotes the covariant derivative, K is the constant sectional curvature of the manifold, and s is considered as a measure of time. When the configuration space is two-dimensional, the scalar curvature reads as $K = \frac{1}{2}\mathcal{R}$ with

$$\mathcal{R} = \frac{(\nabla V)^2}{(E-V)^3} + \frac{\Delta V}{(E-V)^2}, \tag{3}$$

where ∇, Δ and E stand, respectively, for the Euclidean gradient, Laplacian operators and energy. By a projection along the direction normal to the geodesic, covariant derivatives become ordinary derivatives, i.e., $D/ds \equiv d/ds$. The geodesic flow is unstable only if $K < 0$. This means the onset of chaos in the case of compact manifolds. On the other hand, the geodesic flow is stable if $K > 0$. This instability criterion by negative curvatures is constructed on the Jacobi metric. Hereafter the criterion is labeled as C1.

It is worth stressing that this criterion C1 should be carefully used when the curvature K is no longer a constant. It is still suitable for the case that the nonconstant curvature K is everywhere negative. In fact, the case corresponds to that of hyperbolicity, viewed as an important mechanism leading to the origin of the instability of the geodesics. However, C1 is somewhat questionable to treat the manifolds whose curvature is neither constant nor everywhere negative. In practice, it was found that chaos can be caused not only by negative curvatures but also by positive nonconstant curvatures [6,9–11]. In other words, a negative curvature is not necessary at all for the presence of chaos in a geodesic flow. These facts show sufficiently that besides the mechanism of the hyperbolicity another mechanism inducing chaos in geodesic flows of physical relevance, namely, parametric instability[1] due to the variability of curvature along the geodesics, should be present [10–13]. This mechanism is obviously active also when the mechanical manifold is mainly positively curved. Besides the reason why a fluctuating positive nonconstant curvature along the geodesic can produce instability, these articles gave an analytic formula for the largest Lyapunov exponent depending on the evolution of the averages and fluctuations of the curvature of the configuration space with varying energy. Here are some details. In order to cope with the limitation of C1, Refs. [10,11] replaced the Jacobi equation (2) with the following expression

$$\frac{d^2\psi}{dt^2} + \langle k_R\rangle_\mu \psi + \sigma_\Omega \eta(t)\psi = 0, \tag{4}$$

where ψ denotes any of the components about the Jacobi field $\boldsymbol{J}$ with N dimensions, and η is a Gaussian function with zero mean and unit variance. In addition, the average Ricci curvature and its fluctuation are respectively written as

$$\Omega_0 = \langle k_R\rangle_\mu = \frac{1}{N-1}\langle \Delta V\rangle_\mu, \tag{5}$$

$$\sigma_\Omega^2 = \frac{1}{N-1}\langle \delta^2 K_R\rangle_\mu = \frac{1}{N-1}[\langle(\Delta V)^2\rangle_\mu - \langle \Delta V\rangle_\mu^2], \tag{6}$$

where $\langle\rangle_\mu$ stands for static averages computed with the microcanonical measure μ on the constant energy surface of phase space. In a word, both of them are functions of the energy E. They also determine the largest Lyapunov exponent

$$\lambda(\Omega_0, \sigma_\Omega, \tau) = \frac{1}{2}\left(\Lambda - \frac{4\Omega_0}{3\Lambda}\right) \tag{7}$$

with

$$\Lambda = \left\{2\tau\sigma_\Omega^2 + \left[\left(\frac{4\Omega_0}{3}\right)^3 + (2\tau\sigma_\Omega^2)^2\right]^{1/2}\right\}^{1/3}, \tag{8}$$

$$2\tau = \frac{\pi\Omega_0^{1/2}}{2[\Omega_0(\Omega_0+\sigma_\Omega)]^{1/2} + \pi\sigma_\Omega}. \tag{9}$$

Of course, λ is still a function of the energy E. Hence the criterion marked as C2, by applying curvature fluctuations of the manifolds to find the energy dependence of the geometric instability exponent, is constructed. Some examples have displayed that it is more sensitive to detect instability or chaos than the method C1. Readers are also recommended to see a thorough discussion about similar topics which is given in the recently published book entitled "Geometry and Topology in Hamiltonian Dynamics and Statistical Mechanics" by Pettini [14]. Additionally, it is worth mentioning that although several examples in [10,11] described that analytic results of Eq. (7) for the largest Lyapunov exponent vs the energy density coincide basically with numeric results, the Lyapunov exponent by Eq. (7) is quite different from the usual sense of the Lyapunov exponent in the known literature (e.g. see [15]). The Lyapunov exponent from Eq. (7), as a function of the energy, is of help for telling one which energy is possible or impossible to bring chaos, but it is very difficult to provide any details about the

[1] It means that parameters vary periodically or quasiperiodically in time.