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Explicit Calabi–Yau metrics in $D = 6$ possessing an isometry group with orbits of codimension one

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ABSTRACT

A method for constructing explicit Calabi–Yau metrics in six dimensions with an isometry group with orbits of codimension one is presented. The equations to solve are nonlinear, but become linear when certain geometrical objects defining the metric vary over a complex submanifold. It is shown that this method encode known examples such as the CY metrics of [G. Gibbons, H. Lu, C. Pope, K. Stelle, Nuclear Phys. B 623 (2002) 3] or the asymptotic form of the BKTY metrics of [S. Bando, R. Kobayashi, Math. Ann. 287 (1990) 175; Proc. 21st Int. Taniguchi Symp, Lecture Notes in Pure Math. 1339 (1988) 20] and [G. Gibbons, P. Rychenkova, J. Geom. Phys. 32 (2000) 311], but we construct new ones.

1. Introduction

The development of the subject of Calabi–Yau (CY) manifolds is an illustrative example of the interplay between algebraic geometry and string theory. On one hand, CY spaces are interpreted as internal spaces of string and M-theory giving supersymmetric field theories after compactification. In fact, CY 3-folds may provide compactifications which are more realistic than the ones corresponding to other Ricci flat manifolds such as G_2 holonomy spaces, for which the generation of chiral matter and non-abelian gauge symmetries seems harder (but not impossible) to achieve. On the other hand, string theory compactifications stimulated several new trends in the algebro-geometrical aspects of CY spaces, an example is the subject of mirror symmetry.

By definition a CY manifold is a compact Kahler n-dimensional manifold with vanishing first Chern class. The Yau proof of the Calabi conjecture implies that these manifolds admit a Ricci flat metric and their holonomy is reduced from SO($2n$) to SU(n) [1]. Although these Ricci flat metrics exist, no explicit expression has been found. For the non-compact case, the definition usually adopted is that a CY manifold is a Ricci flat Kahler manifold, which also implies that the holonomy is reduced to SU(n) or to a smaller subgroup. In this case, several explicit metrics have been constructed. One of the oldest examples are the asymptotically conical metrics presented in [2]. Another interesting metrics have been found in [3] and [4]. The last ones have curvature singularities, but for some of them the contribution to the gravitational action is finite and this makes plausible that they can be extended to a regular gravitational instanton in six dimensions. In fact, some of these solutions were identified as the asymptotic form of the generalized Bando–Kobayashi–Tian–Yau (BKTY) metrics [5] and [6], which are by construction Calabi–Yau.

The CY metrics briefly described above are six-dimensional and possess an isometry group with orbits of codimension one, and some of them arise as fibrations over hyperkahler gravitational instantons. The present work deals with this type of solutions. The goal we have achieved is to find a method for constructing CY metrics that include the ones of [3,4] as a particular case. But we are also able to construct different type of examples, which are non-compact regular metrics fibered over the Eguchi–Hanson gravitational instanton.

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The organization of the paper is the following. In Section 2 a large family of CY six metrics with isometry groups of codimension one is characterized in terms of a complicated nonlinear system of equations. The result is stated in a proposition, in order to make the exposition clear. We show that the family we are describing encode the CY metrics of [3] and the asymptotic form of certain BKTY metrics as particular cases. In Section 3 a family of solutions of that system is parameterized in terms of an initial hyperkahler structure. It is also shown that the nonlinear system becomes linear when the objects describing the 6-metric are defined over complex submanifolds of the initial hyperkahler manifold. Arguably, this family of metrics is not of the same type than the ones presented in [3]. In Section 4 we construct some incomplete examples which exemplify our solution generating technique. In addition, we construct a family of CY metrics by taking the Eguchi–Hanson gravitational instanton as the initial hyperkahler structure and by assuming that the objects defining the metric are varying on an specific complex submanifold. The resulting system is linearized and the resulting CY metrics are non-compact and regular.

2. CY 3-folds with an isometry group of co-dimension one

2.1. The defining equations

In the present section a large family of CY manifolds with an isometry group of co-dimension one is characterized. It is assumed that the Killing vector V corresponding to this isometry preserve not only the metric, but the full SU(3) structure. It will be convenient to remind some elementary definitions about CY manifolds first. As is well known, a CY manifold M_6 with a Ricci flat metric g_6 admits at least one complex structure J for which $g_6(X, JY) = g_6(JX, Y)$ being X and Y arbitrary vector fields, in such a way that the two-form

$$\omega_6 = (g_6)_{\mu\alpha} J^{\alpha}_{\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \tag{2.1}$$

is closed. Here x^{μ} is an arbitrary choice of coordinates for M_6. The SU(3) structure is given by the closed two-form ω_6 and a complex closed 3-form

$$\Psi = \psi_+ + \mathrm{i}\psi_- \tag{2.2}$$

of type $(3, 0)$ with respect to J, satisfying the following compatibility conditions [7]

$$\omega_6 \wedge \psi_{\pm} = 0, \qquad \psi_+ \wedge \psi_- = \frac{2}{3}\omega_6 \wedge \omega_6 \wedge \omega_6 = \frac{2}{3}V(g_6), \qquad \psi_+ \wedge \psi_+ = \psi_- \wedge \psi_- = 0. \tag{2.3}$$

In the last expression $V(g_6)$ denote the volume form of g_6. The knowledge of the SU(3) structure is enough to determine the metric g_6 and if it is closed, then g_6 is a Ricci flat Kahler and with holonomy in SU(3). The converse of these statements are also true, that is, for any Ricci flat Kahler metric there will exist an SU(3) structure satisfying (2.3) and also

$$\mathrm{d}\omega_6 = \mathrm{d}\psi_+ = \mathrm{d}\psi_- = 0. \tag{2.4}$$

Furthermore, if there is a Killing vector V for g_6 then there exists a local coordinate system (α, x^i) with $i = 1, \ldots, 5$ for which $V = \partial_\alpha$ and for which the metric tensor g_6 take the following form

$$g_6 = \frac{(\mathrm{d}\alpha + A)^2}{H^2} + Hg_5. \tag{2.5}$$

In the last expression the function H, the one form A and the metric tensor g_5 are independent on the coordinate α, thus these objects live in a 5-dimensional space which we denote M_5. It is convenient to express the metric g_5 appearing in (2.5) in tetrad form as $g_5 = e^a \otimes e^a$ with $a = 1, \ldots, 5$ for some basis e^a. Then, if V also preserve the $SU(3)$ structure (as we are assuming) one has the decomposition

$$\omega_6 = \omega_1 + \frac{1}{\sqrt{H}} e_5 \wedge (\mathrm{d}\alpha + A), \tag{2.6}$$

$$\psi_+ = H^{3/2}\omega_3 \wedge e^5 + \omega_2 \wedge (\mathrm{d}\alpha + A), \qquad \psi_- = -H^{3/2}\omega_2 \wedge e^5 + \omega_3 \wedge (\mathrm{d}\alpha + A). \tag{2.7}$$

Here e^5 is a one-form that will be determined by the closure of the SU(3) structure (2.4) and ω_i is a triplet of 2-forms which are expressed as $\omega_i = e^4 \wedge e^i + \epsilon_{ijk} e^j \wedge e^k$. By assumption V preserve (2.6) and (2.7) and therefore e^5 and ω_i do not depend on the coordinate α, i.e, they are also defined in M_5.

The task now is to derive the consequences of the CY relations (2.3) and (2.4) for the generic ansatz (2.5)–(2.7). From (2.3) it follows immediately that

$$\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_3 = \omega_2 \wedge \omega_3 = 0 \tag{2.8}$$

$$\omega_1 \wedge \omega_1 = H^2 \omega_2 \wedge \omega_2 = H^2 \omega_3 \wedge \omega_3. \tag{2.9}$$