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One-sided invariant manifolds, recursive folding, and curvature singularity in area-preserving nonlinear maps with nonuniform hyperbolic behavior

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Abstract

Two-dimensional nonlinear models of conservative dynamics are typically nonuniformly hyperbolic in that there are nonhyperbolic trajectories that coexist with a "massive" hyperbolic region. We investigate the influence of nonhyperbolic points on the global geometric structure of a invariant manifolds associated with points of the hyperbolic region. As a case study, we consider a transformation of the Standard Map family and analyze the structure of invariant manifolds in the neighborhood of an isolated parabolic (fixed) point $\mathbf{x}_p$. This analysis shows the existence of lobes enclosing the parabolic point, that is, of simply connected regions containing $\mathbf{x}_p$ whose boundary is formed by two continuous arcs of stable and unstable manifolds that intersect only at two points. From the existence of such regions, we derive that (i) there are points of the hyperbolic region where the local curvature of invariant manifolds is arbitrarily large and (ii) manifolds possess the recursively folding property. Property (ii) means that given an invariant manifold $\mathscr{W}$ and established an orientation on it, in the neighborhood of any point of the chaotic region there are nearby arcs of $\mathscr{W}$ that are traveled in opposite directions. We propose an archetypal model for which the existence of lobes and the recursive folding property can be derived analytically. The impact of nonuniform hyperbolicity on the evolution of physical processes that occur along with phase space mixing is also addressed.

1. Introduction

The theoretical understanding and interpretation of chaos in low-dimensional conservative dynamics is still largely based on the paradigm of uniformly hyperbolic systems (also referred to as *Anosov systems* [1]) that was laid out in the 60's by Anosov [2] and Smale [3]. Shortly, a uniformly hyperbolic transformation $\mathbf{f}$ of a smooth manifold $\mathscr{M}$. is a measure-preserving map for which there exists, *at any point* $\mathbf{x}$ of $\mathscr{M}$, a splitting of the tangent space $T_\mathbf{x} = E^u_\mathbf{x} \oplus E^s_\mathbf{x}$, invariant under the differential $D\mathbf{f}$ of transformation, that is, $D\mathbf{f}|_\mathbf{x}(E^\alpha_\mathbf{x}) = E^\alpha_{\mathbf{f}(\mathbf{x})}$, where $\alpha = s, u$. The splitting is uniquely identified by the property that the norm of vectors $\mathbf{v}$ ($\mathbf{w}$) belonging to the subspace $E^u_\mathbf{x}(E^s_\mathbf{x})$ shrinks exponentially fast to zero for

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negative (positive) times under the differential of the transformation, i.e. $\|D\mathbf{f}^{-n}|_{\mathbf{x}} \cdot \mathbf{v}\| < A\lambda^n \|\mathbf{v}\|$, and $\|D\mathbf{f}^{n}|_{\mathbf{x}} \cdot \mathbf{w}\| < A\lambda^n \|\mathbf{w}\|$, where $A > 0$ and $\lambda < 1$ for any $\mathbf{x} \in \mathscr{M}$ (the wording *uniformly* hyperbolic indicates precisely that the constants A and λ entering the bounds of vectors norm dynamics hold for the entire manifold $\mathscr{M}$). Uniformly hyperbolic systems possess "nice" and interesting features that can be proved rigorously such as positivity of topological and metric entropies, mixing property, existence of global stable and unstable invariant manifolds that are dense in $\mathscr{M}$. and everywhere transverse to each other, shadowing property, e.g. ensuring that there exist trajectories of the system ε-close to (computer-generated) pseudotrajectories [4]. Classical archetypal examples of Anosov systems are hyperbolic toral automorphisms, which are represented by the action of hyperbolic matrices with integer entries and unit modulus determinant on the unit two-dimensional torus [1].

However, the Anosov paradigm can account solely for some of the dynamical features of generic nonlinear systems that are considered physically interesting. In generic nonlinear systems, the most evident indication of departure from uniform hyperbolic behavior is that chaos is generally "massive but not ubiquitous", meaning that there exists points of the phase space with zero Lyapunov exponent (e.g. elliptic or parabolic periodic points or quasiperiodic trajectories) that coexist with a set of positive measure of hyperbolic trajectories. This picture agrees with the categorization introduced by Pesin in the mid 70s [5], who defined the class of *nonuniformly hyperbolic systems*. In a two-dimensional phase space, nonuniformly hyperbolic systems are defined as those systems for which the set of all hyperbolic trajectories (referred to as *the Pesin set*, say $\mathscr{P}$) has positive measure. The presence of nonhyperbolic orbits can influence geometric and statistical properties associated with points of the Pesin set $\mathscr{P}$. For instance, the existence of a strictly positive constant A entering the bounds of vectors norm dynamics is granted only pointwise (i.e. one cannot find a constant that works for the entire set $\mathscr{P}$), and so is the lowerbound for the angle between stable and unstable invariant directions. The latter property implies that in the Pesin region of a nonuniformly hyperbolic system, it is possible to find points at which the angle between stable and unstable directions becomes arbitrarily small, i.e. the stable and unstable manifolds become almost tangent to each other.

In the physics literature, systems that exhibit invariant manifold tangencies have also been referred to as *nonhyperbolic* [6]. In this article, we stick to Pesin's categorization as we find that it could be misleading to use the adjective nonhyperbolic for systems that show hyperbolic behavior in a set of points of positive measure, possibly even of full measure (In Section 4, we provide an archetypal example of nonuniform hyperbolic system for which nonhyperbolic orbits form a set of zero Lebesgue measure).

The focus of this article is to analyze how the presence of nonhyperbolic points influences the local and global geometry of invariant manifolds associated with points of the Pesin set $\mathscr{P}$ Specifically, we show that the presence of an isolated parabolic point that lies at the boundary of $\mathscr{P}$ implies that there are points of the Pesin region where stable and unstable manifolds possess arbitrarily high curvature.

Furthermore, the analysis of the local structure of the stable and unstable manifolds at, and in the vicinity of, the parabolic point allows us to derive a geometric property of invariant manifolds, referred to as *recursive folding*, which we believe to be typical of physically interesting nonlinear models, but which cannot occur in strictly uniformly hyperbolic area-preserving systems.

2. Case study

As an illustrative example of nonuniformly hyperbolic system, we consider the Standard Map (SM) family [7]

$$\mathbf{f}_\tau(\mathbf{x}) = \begin{cases} x + y + \tau\sin(2\pi x) \\ y + \tau\sin(2\pi x) \end{cases} \pmod 1, \tag{1}$$

where $(x, y) = \mathbf{x}$ are coordinates on the unit square interval $[0,1) \times [0,1)$ representing a global projection chart of the two-torus (the coordinates of the image point are considered "mod 1"), and τ is a real parameter. It is well established that the maps of this family display the basic phenomenology that characterizes conservative chaotic dynamics of two-dimensional physically interesting nonlinear systems [8], including stroboscopic maps of physically realizable time-periodic incompressible flows in closed domains [9]. Specifically, we consider the map $\mathbf{f}_1(\mathbf{x})$ corresponding to the value $\tau = 1$, whose phase space is characterized by a main chaotic region, henceforth referred to as the Pesin set $\mathscr{P}$, that invades the entire torus with the exception of two small elliptic islands centered around the period-two elliptic orbit constituted by the two points (1/4,1/2) and (3/4,1/2) (see Fig. 1(A)). Fig. 1(B) shows the structure of the unstable manifold, $\mathscr{W}_0^u$, of the origin (an hyperbolic fixed point for the map $\mathbf{f}_1(\mathbf{x})$), computed by tracking for $n = 7$ periods a small segment aligned with the local unstable direction at the point. Likewise those associated with hyperbolic toral homeomorphism, the manifold $\mathscr{W}_0^u$ is dense in the chaotic region, which, however, in the present case does not coincide with the entire manifold. On the other hand, the geometry of $\mathscr{W}_0^u$ is different and remarkably more complicated than that of invariant