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A new Lie algebra, a corresponding multi-component integrable hierarchy and an integrable coupling ☆

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Abstract

A new simple loop algebra is constructed, which is devote to establishing an isospectral problem. By making use of Tu scheme, a new multi-component integrable hierarchy is obtained. Again via expanding the loop algebra above, another higher-dimensional loop algebra is presented. It follows that the binary integrable coupling systems are given. This method proposed in this paper can be used to other soliton hierarchies.

1. Introduction

The Lax pair method [1] is a general technique for generating single component integrable hierarchies of soliton equations. In terms of the Lax equation

$$L_t = AL - LA \equiv [A, L], \tag{1}$$

many interesting integrable soliton equations with physical backgrounds, such as the KdV equation, the Burgers equations and so on [2,3] have been worked out. Tu Guizhang further developed the Lax pair method and presented a simple and efficient approach for generating integrable Hamiltonian hierarchies [4], Ma Wenxiu called it Tu scheme [5]. By taking advantage of the scheme, the celebrated hierarchies, such as AKNS hierarchy, KN hierarchy, WKI hierarchy, etc. were given [4–9]. Hu Xingbiao extended the Tu scheme in the frame of the loop algebra $\widetilde{A}_2$ and obtained some interesting results [6,7]. Guo Fukui again presented some integrable Hamiltonian systems of soliton equations with multi-potential functions, in general, less than 8 functions, by changing the power times of the spectral parameters in the frame of loop algebra $\widetilde{A}_1$ [10,11]. By using Guo's idea, a few interesting results were also obtained in [12,13]. As far as the multi-component integrable hierarchies of soliton equations are concerned, there have been developments in [14,15]. Ma Wenxiu and Zhou Ruguang gave the multi-component AKNS hierarchy by use of the Lax pair method [16]. In order to conveniently produce some interesting multi-component integrable hierarchies, a simple method was proposed in [17]. As its application, the multi-component system M-AKNS-KN hierarchy was generated. In this

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paper, a simple and convenient way for obtaining multi-component integrable systems is presented. Firstly, a new Lie algebra is constructed, whose commuting operation is just the same with that in [17]. Secondly, a corresponding higher-dimensional loop algebra $\widetilde{G}_M$ is showed. It follows that an isospectral problem is established. By employing the Tu scheme, a new multi- component integrable hierarchy is obtained.

In addition, as we know, integrable couplings are a quite important aspect in the field of soliton theory [18]. A general method is the perturbation approach. In [19], a simple way for producing integrable couplings were once given. By employing this approach, the integrable couplings of some known integrable systems were obtained [19–22]. Therefore, in this paper, we also construct another higher-dimensional loop algebra to deduce the integrable coupling of the hierarchy (17) presented below.

2. A new Lie algebra and its corresponding loop algebra

In [17], we presented the following definition: Set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^T, \beta = (\beta_1, \beta_2, \ldots, \beta_M)^T$ to be two vectors, define their product $\alpha * \beta$ as follows

$$\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \ldots, \alpha_M\beta_M)^T.$$

Thus, a Lie algebra was constructed by

$$G_M = \{a = (a_{ij})_{M\times 3} = (a_1, a_2, a_3)\}, \tag{2}$$

with a commuting operation defined as

$$[a, b] = (a_2 * b_3 - a_3 * b_2, 2(a_1 * b_2 - a_2 * b_1), 2(a_3 * b_1 - a_1 * b_3)), \forall_{a,b} \in G_M. \tag{3}$$

A corresponding loop algebra is given by

$$\widetilde{G}_M = \{a\lambda^m, a \in G_M, m = 0, \pm 1, \pm 2, \ldots,\} \tag{4}$$

with a commuting operation expressed as

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \forall a, b \in G_M. \tag{5}$$

We find that it is not easy to directly use the loop algebra $\widetilde{G}_M$ to work out multi-component integrable hierarchies. In order to overcome the shortcoming, we construct another Lie algebra.

Definition 1. Let $I_M = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{M\times 1}$ be a matrix, and set

$$h = (I_M, 0, 0), e = (0, I_M, 0), f = (0, 0, I_M), \tag{6}$$

where M is a positive integer. A commuting relation among them is defined as

$$[h, e] = -[e, h] = 2e, [h, f] = -[f, h] = -2f, [e, f] = -[f, e] = h. \tag{7}$$

Then $\{h,e,f\}$ along with (7) constitutes a Lie algebra, and we denote it as G_M.

Definition 2. If $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{pmatrix}$ is a column vector, $A = (0, \ldots, 0, I_M, 0, \ldots, 0)_{M\times N}$ is a $M \times N$ matrix, where I_M is in the ith column of the matrix a. Then a commuting relation between them is defined as

$$\alpha \cdot A = A \cdot \alpha = (0, \ldots, \alpha * I_M, 0, \ldots, 0). \tag{8}$$

Let $\alpha = a_1 \cdot h + a_2 \cdot e + a_3 \cdot f$ and $b = b_1 \cdot h + b_2 \cdot e + b_3 \cdot f$, then from the loop algebra (6) and the definition (8), we have

$$\begin{aligned}[a, b] &= (a_2 * b_3 - a_3 * b_2) \cdot h + 2(a_1 * b_2 - a_2 * b_1) \cdot e + 2(a_3 * b_1 - a_1 * b_3) \cdot f \\ &= (a_2 * b_3 - a_3 * b_2, 2(a_1 * b_2 - a_2 * b_1), 2(a_3 * b_1 - a_1 * b_3)),\end{aligned} \tag{9}$$

which is just the formula (3). where $a_i = (a_{m1}^{(i)}, a_{m2}^{(i)}, \ldots, a_{mM}^{(i)})^T, b_i = (b_{m1}^{(i)}, b_{m2}^{(i)}, \ldots, b_{mM}^{(i)})^T, i = 1, 2, 3$. In what follows, we shall find that the Lie algebra (6) is more convenient than G_M in the aspect of deducing multi-component integrable hierarchies.